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**MECANICĂ
MATEMATICĂ**

1985

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SUR L'INVARIABILITÉ DE L'AXE INSTANTANÉ DE ROTATION

II. LE CAS DES DEUX REPÈRES CARTÉSIENS TRIORTHOGONAUX QUI SE MEUVENT EXCLUSIVEMENT PAR UNE TRANSLATION L'UN PAR RAPPORT À L'AUTRE

PAR

C. ACKER et ELENA ACKER

Considérons le repère $X_1O_1Y_1Z_1$, cartésien triorthogonal, solidaire d'un corps rigide et aussi le repère $X_2O_2Y_2Z_2$, cartésien triorthogonal, ayant un mouvement quelconque par rapport au premier repère (v. fig. 1). Alors les expressions analytiques des vecteurs unités $\mathbf{i}_2, \mathbf{j}_2, \mathbf{k}_2$ des axes des coordonnées du repère $X_2O_2Y_2Z_2$, par rapport au repère $X_1O_1Y_1Z_1$, sont

$$(1) \quad \begin{cases} \mathbf{i}_2 = \alpha_{11}\mathbf{i}_1 + \alpha_{12}\mathbf{j}_1 + \alpha_{13}\mathbf{k}_1, \\ \mathbf{j}_2 = \alpha_{21}\mathbf{i}_1 + \alpha_{22}\mathbf{j}_1 + \alpha_{23}\mathbf{k}_1, \\ \mathbf{k}_2 = \alpha_{31}\mathbf{i}_1 + \alpha_{32}\mathbf{j}_1 + \alpha_{33}\mathbf{k}_1; \end{cases}$$

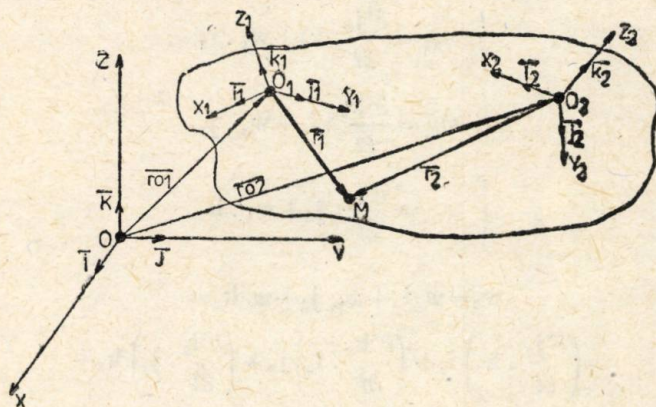


Fig. 1

ici α_{mn} ($m, n=1, 2, 3$) sont les cosinus directeurs des vecteurs unités $\mathbf{i}_2, \mathbf{j}_2, \mathbf{k}_2$ et pendant le mouvement quelconque de $X_2O_2Y_2Z_2$, par rapport au repère $X_1O_1Y_1Z_1$, ces cosinus directeurs dépendent effectivement du temps t (en général α_{mn} n'ont pas des valeurs constantes pendant l'écoulement du temps t ; $\alpha_{mn} = \text{const.}$ lorsque les deux repères sont solidaires l'un de l'autre, par exemple, lorsque les deux repères sont solidaires d'un même corps rigide (v. [6])).

Donc les dérivées absolues des vecteurs unités $\mathbf{i}_2, \mathbf{j}_2, \mathbf{k}_2$, par rapport au temps t , sont

$$(2) \quad \begin{cases} \dot{\mathbf{i}}_2 = \frac{\partial^* \mathbf{i}_2}{\partial t} + \mathbf{w}_1 \times \mathbf{i}_2, \\ \dot{\mathbf{j}}_2 = \frac{\partial^* \mathbf{j}_2}{\partial t} + \mathbf{w}_1 \times \mathbf{j}_2, \\ \dot{\mathbf{k}}_2 = \frac{\partial^* \mathbf{k}_2}{\partial t} + \mathbf{w}_1 \times \mathbf{k}_2, \end{cases}$$

ce qui implique

$$(3) \quad \begin{cases} w_{x_2} = \mathbf{j}_2 \cdot \mathbf{k}_2 = \left(\frac{\partial^* \mathbf{j}_2}{\partial t} + \mathbf{w}_1 \times \mathbf{j}_2 \right) \cdot \mathbf{k}_2 = \frac{\partial^* \mathbf{j}_2}{\partial t} \cdot \mathbf{k}_2 + \mathbf{k}_2 \cdot (\mathbf{w}_1 \times \mathbf{j}_2), \\ w_{y_2} = \mathbf{k}_2 \cdot \mathbf{i}_2 = \left(\frac{\partial^* \mathbf{k}_2}{\partial t} + \mathbf{w}_1 \times \mathbf{k}_2 \right) \cdot \mathbf{i}_2 = \frac{\partial^* \mathbf{k}_2}{\partial t} \cdot \mathbf{i}_2 + \mathbf{i}_2 \cdot (\mathbf{w}_1 \times \mathbf{k}_2), \\ w_{z_2} = \mathbf{i}_2 \cdot \mathbf{j}_2 = \left(\frac{\partial^* \mathbf{i}_2}{\partial t} + \mathbf{w}_1 \times \mathbf{i}_2 \right) \cdot \mathbf{j}_2 = \frac{\partial^* \mathbf{i}_2}{\partial t} \cdot \mathbf{j}_2 + \mathbf{j}_2 \cdot (\mathbf{w}_1 \times \mathbf{i}_2), \end{cases}$$

donc les projections orthogonales du vecteur vitesse angulaire $\mathbf{w}_2$ appartenant au repère $X_2O_2Y_2Z_2$, sur les axes du repère $X_2O_2Y_2Z_2$, lorsque le mouvement du repère $X_2O_2Y_2Z_2$ est regardé par un observateur solidaire du repère fixe ont aussi les expressions

$$(4) \quad \begin{cases} w_{x_2} = \frac{\partial^* \mathbf{j}_2}{\partial t} \cdot \mathbf{k}_2 + \mathbf{w}_1 \cdot \mathbf{i}_2, \\ w_{y_2} = \frac{\partial^* \mathbf{k}_2}{\partial t} \cdot \mathbf{i}_2 + \mathbf{w}_1 \cdot \mathbf{j}_2, \\ w_{z_2} = \frac{\partial^* \mathbf{i}_2}{\partial t} \cdot \mathbf{j}_2 + \mathbf{w}_1 \cdot \mathbf{k}_2, \end{cases}$$

c'est à dire

$$(5) \quad \begin{aligned} \mathbf{w}_2 &= w_{x_2} \mathbf{i}_2 + w_{y_2} \mathbf{j}_2 + w_{z_2} \mathbf{k}_2 = \\ &= \left(\frac{\partial^* \mathbf{j}_2}{\partial t} \cdot \mathbf{k}_2 \right) \mathbf{i}_2 + \left(\frac{\partial^* \mathbf{k}_2}{\partial t} \cdot \mathbf{i}_2 \right) \mathbf{j}_2 + \left(\frac{\partial^* \mathbf{i}_2}{\partial t} \cdot \mathbf{j}_2 \right) \mathbf{k}_2 + \\ &\quad + (\mathbf{w}_1 \cdot \mathbf{i}_2) \mathbf{i}_2 + (\mathbf{w}_1 \cdot \mathbf{j}_2) \mathbf{j}_2 + (\mathbf{w}_1 \cdot \mathbf{k}_2) \mathbf{k}_2, \end{aligned}$$

donc

$$(6) \quad \mathbf{w}_2 = \mathbf{w}_{21} + \mathbf{w}_1,$$

où $\mathbf{w}_{21}$ est le vecteur vitesse angulaire du repère $X_2O_2Y_2Z_2$ (et de tous les points solidaires de ce repère), par rapport au repère $X_1O_1Y_1Z_1$, c'est à dire pour l'observateur solidaire du repère $X_1O_1Y_1Z_1$ qui regarde le mouvement du repère $X_2O_2Y_2Z_2$ (et cela parce que les dérivées $\partial^* \mathbf{j}_2 / \partial t$, $\partial^* \mathbf{k}_2 / \partial t$, $\partial^* \mathbf{i}_2 / \partial t$, de plus haut, sont relatives au repère $X_1O_1Y_1Z_1$ et non pas

relatives au repère fixe), tandis que w_1 est le vecteur vitesse angulaire du repère $X_1O_1Y_1Z_1$ (et de tous les points solidaires de ce repère) par rapport au repère fixe, donc pour l'observateur solide du repère fixe qui regarde le mouvement du repère $X_1O_1Y_1Z_1$.

Par de similaires considérations, mais en utilisant les expressions analytiques des vecteurs unités i_1, j_1, k_1 des axes du repère $X_1O_1Y_1Z_1$ par rapport au repère $X_2O_2Y_2Z_2$, on obtient aussi

$$(7) \quad w_1 = w_{12} + w_2,$$

où w_{12} est le vecteur vitesse angulaire du repère $X_1O_1Y_1Z_1$ (et de tous les points solidaires de ce repère) par rapport au repère $X_2O_2Y_2Z_2$, c'est à dire pour l'observateur solide du repère $X_2O_2Y_2Z_2$ qui regarde le mouvement du repère $X_1O_1Y_1Z_1$.

Les relations (6) et (7) impliquent

$$(8) \quad w_{12} = -w_{21}.$$

Lorsque le repère $X_2O_2Y_2Z_2$ se meut, par rapport au repère $X_1O_1Y_1Z_1$ (v. fig. 1), exclusivement par une translation, alors les cosinus directeurs α_{mn} ($m, n=1, 2, 3$), dans les relations (1), deviennent constants pendant l'écoulement du temps t , ce qui implique les valeurs nulles des dérivées des vecteurs unités i_2, j_2, k_2 , relatives au repère $X_1O_1Y_1Z_1$, c'est à dire

$$(9) \quad \frac{\partial^* i_2}{\partial t} = \frac{\partial^* j_2}{\partial t} = \frac{\partial^* k_2}{\partial t} = 0,$$

ce qui implique (v. les relations (5)–(8))

$$(10) \quad w_{21} = -w_{12} = 0.$$

Donc lorsque le repère $X_2O_2Y_2Z_2$, cartésien triorthogonal, se meut exclusivement par une translation, par rapport au repère $X_1O_1Y_1Z_1$, cartésien triorthogonal, alors les respectifs vecteurs vitesses angulaires, correspondants aux mouvements de ces repères par rapport au repère fixe, sont égaux (v. (6), (7), (10))

$$(11) \quad w_1 = w_2,$$

les respectifs axes intantanés de rotation des deux repères $X_1O_1Y_1Z_1, X_2O_2Y_2Z_2$, correspondants aux mouvements de ces repères par rapport au repère fixe, ayant une même direction (ces axes étant donc parallèles).

Nous allons démontrer au dessous que les deux axes intantanés de rotation ont, dans ce cas, pourtant des positions distinctes, en général.

Considérons donc la vitesse de l'origine O_2 du repère $X_2O_2Y_2Z_2$, — qui se meut exclusivement par une translation, par rapport au repère $X_1O_1Y_1Z_1$ —, en utilisant le repère $X_1O_1Y_1Z_1$ (solidaire du corps rigide) pour exprimer cette vitesse

$$(12) \quad v_{O_2} = v_{O_1} + w_1 \times \overrightarrow{O_1O_2} + \frac{\partial^* \overrightarrow{O_1O_2}}{\partial t}$$

($\partial^* \overrightarrow{O_1O_2} / \partial t$ est la dérivée de $\overrightarrow{O_1O_2}$ relative au repère $X_1O_1Y_1Z_1$), donc

$$(13) \quad v_{O_1} - v_{O_2} = w_1 \times \overrightarrow{O_1O_2} + \frac{\partial^* \overrightarrow{O_1O_2}}{\partial t}$$

et alors la différence

$$(14) \quad \mathbf{D} = \left(\mathbf{r}_{O_2} + \frac{\mathbf{w}_2 \times \mathbf{v}_{O_2}}{w_2^2} \right) - \left(\mathbf{r}_{O_1} + \frac{\mathbf{w}_1 \times \mathbf{v}_{O_1}}{w_1^2} \right)$$

(en tenant compte de (11)) devient

$$(15) \quad \mathbf{D} = (\mathbf{r}_{O_2} - \mathbf{r}_{O_1}) + \frac{\mathbf{w}_1}{w_1^2} \times (\mathbf{v}_{O_2} - \mathbf{v}_{O_1})$$

et en substituant $\mathbf{r}_{O_2} - \mathbf{r}_{O_1}$ par $\overrightarrow{O_1 O_2}$ (v. fig. 1) et $\mathbf{v}_{O_2} - \mathbf{v}_{O_1}$ par l'expression préparée plus haut, on a

$$(16) \quad \mathbf{D} = \overrightarrow{O_1 O_2} + \frac{\mathbf{w}_1}{w_1^2} \times (\mathbf{w}_1 \times \overrightarrow{O_1 O_2}) + \frac{\mathbf{w}_1}{w_1^2} \times \frac{\partial^* \overrightarrow{O_1 O_2}}{\partial t},$$

c'est à dire

$$(17) \quad \mathbf{D} = \overrightarrow{O_1 O_2} + \mathbf{w}_1 \left(\frac{\mathbf{w}_1}{w_1^2} \cdot \overrightarrow{O_1 O_2} \right) - \overrightarrow{O_1 O_2} + \frac{\mathbf{w}_1}{w_1^2} \times \frac{\partial^* \overrightarrow{O_1 O_2}}{\partial t},$$

donc

$$(18) \quad \mathbf{D} = \mathbf{w}_1 \left(\frac{\mathbf{w}_1}{w_1^2} \cdot \overrightarrow{O_1 O_2} \right) + \frac{\mathbf{w}_1}{w_1^2} \times \frac{\partial^* \overrightarrow{O_1 O_2}}{\partial t}.$$

La relation évidente (v. (14))

$$(19) \quad \mathbf{r}_{O_2} + \frac{\mathbf{w}_2 \times \mathbf{v}_{O_2}}{w_2^2} = \mathbf{r}_{O_1} + \frac{\mathbf{w}_1 \times \mathbf{v}_{O_1}}{w_1^2} + \mathbf{D}$$

devient alors

$$(20) \quad \mathbf{r}_{O_2} + \frac{\mathbf{w}_2 \times \mathbf{v}_{O_2}}{w_2^2} = \mathbf{r}_{O_1} + \frac{\mathbf{w}_1 \times \mathbf{v}_{O_1}}{w_1^2} + \mathbf{w}_1 \left(\frac{\mathbf{w}_1}{w_1^2} \cdot \overrightarrow{O_1 O_2} \right) + \frac{\mathbf{w}_1}{w_1^2} \times \frac{\partial^* \overrightarrow{O_1 O_2}}{\partial t}.$$

Par cela, l'équation vectorielle de l'axe instantané de rotation du repère $X_2 O_2 Y_2 Z_2$, par rapport au repère fixe $XOYZ$ (v. fig. 1), c'est à dire

$$(21) \quad \mathbf{r} = \mathbf{r}_{O_1} + \frac{\mathbf{w}_2 \times \mathbf{v}_{O_2}}{w_2^2} + \lambda \mathbf{w}_2,$$

pourra être écrite aussi (v. (11) et (20))

$$(22) \quad \mathbf{r} = \mathbf{r}_{O_1} + \frac{\mathbf{w}_1 \times \mathbf{v}_{O_1}}{w_1^2} + \left(\frac{\mathbf{w}_1}{w_1^2} \times \frac{\partial^* \overrightarrow{O_1 O_2}}{\partial t} \right) + \left(\frac{\mathbf{w}_1}{w_1^2} \cdot \overrightarrow{O_1 O_2} + \lambda \right) \mathbf{w}_1.$$

Lorsque les repères $X_1 O_1 Y_1 Z_1$, $X_2 O_2 Y_2 Z_2$ sont solidaires d'un même corps rigide (v. [6]), alors la vitesse relative $\partial^* \overrightarrow{O_1 O_2} / \partial t$ du repère $X_2 O_2 Y_2 Z_2$ (qui se meut ici exclusivement par une translation, par rapport au repère $X_1 O_1 Y_1 Z_1$) devient nulle et les axes instantanés de rotation des deux repères, correspondants aux mouvements de ces repères par rapport au repère fixe, coïncident (v. la relation (35) [6]).

Mais, en général, le terme

$$\mathbf{T} = \frac{\mathbf{w}_1}{w_1^2} \times \frac{\partial^* \overrightarrow{O_1 O_2}}{\partial t},$$

du 2-ème membre de la relation (22), n'est pas nul et il indique un déplacement de l'axe instantané de rotation du repère $X_2O_2Y_2Z_2$, dans la direction perpendiculaire sur l'axe instantané de rotation du repère $X_1O_1Y_1Z_1$, par rapport à l'axe instantané de rotation du repère $X_1O_1Y_1Z_1$, lui-même.

Signalons donc la condition générale qui implique la coïncidence des deux axes instantanés de rotation des deux repères $X_1O_1Y_1Z_1$, $X_2O_2Y_2Z_2$, lorsque le repère $X_2O_2Y_2Z_2$ se meut exclusivement par une translation, par rapport au repère $X_1O_1Y_1Z_1$, celle que la vitesse de $X_2O_2Y_2Z_2$, relative au repère $X_1O_1Y_1Z_1$, soit parallèle à l'axe instantané de rotation du repère $X_1O_1Y_1Z_1$, cet axe correspondant au mouvement de $X_1O_1Y_1Z_1$ par rapport au repère fixe $XOYZ$, ce résultat constituant une généralisation d'un théorème antérieur (démontré dans [6]) qui affirmait la coïncidence des axes instantanés de rotation des deux repères $X_1O_1Y_1Z_1$, $X_2O_2Y_2Z_2$ seulement dans le cas d'une translation du repère $X_2O_2Y_2Z_2$ par rapport au repère $X_1O_1Y_1Z_1$, effectuée „à une vitesse relative nulle“ (v. le final de [6]).

Soulignons le caractère nécessaire et suffisant de la condition

$$(23) \quad \frac{\mathbf{w}_1}{w_1^2} \times \frac{\partial^* \vec{O_1O_2}}{\partial t} = 0,$$

pour la coïncidence des deux axes instantanés de rotation; vraiment, la suffisance de la condition (23) est évidente (v. (22) et aussi le final de [6], précisément la relation (35) — (ici on suppose toujours, dès le début, que le repère $X_2O_2Y_2Z_2$ se meut, par rapport au repère $X_1O_1Y_1Z_1$, exclusivement par une translation, ce qui entraîne (11)); la nécessité de la condition (23) résulte aussi facilement; en effet, la coïncidence des deux axes instantanés de rotation implique le fait que les deux axes passent par un même point, du moins, par exemple par le point qui appartient à l'axe instantané de rotation du repère $X_1O_1Y_1Z_1$ et dont le vecteur de sa position est

$$\mathbf{P} = \mathbf{r}_{O_1} + \frac{\mathbf{w}_1 \times \mathbf{v}_{O_1}}{w_1^2},$$

pour obtenir le vecteur $\mathbf{r}$ de la position d'un point quelconque de l'unique axe instantané de rotation des repères $X_1O_1Y_1Z_1$ et $X_2O_2Y_2Z_2$: par exemple pour obtenir le vecteur $\mathbf{r}$ de (22), on doit sommer le vecteur $\mathbf{P}$ de plus haut, avec un vecteur parallèle à $\mathbf{w}_1$, ce qui implique

$$(24) \quad \mathbf{T} \parallel \mathbf{w}_1;$$

l'orthogonalité évidente des vecteurs $\mathbf{T}$, $\mathbf{w}_1$ et leur parallélisme, impliquée de cette façon, entraînent alors (23); en effet, lorsque $\mathbf{T} \perp \mathbf{w}_1$, cela implique la valeur

$$(25) \quad \hat{\alpha} = (2k+1) \frac{\pi}{2} \quad \text{rad.}$$

(k étant un nombre entier quelconque) de l'angle $\hat{\alpha}$ de ces deux vecteurs; alors, (24), qui signifie

$$(26) \quad |\mathbf{T} \times \mathbf{w}_1| = 0,$$

implique, en tenant compte de (25),

$$(27) \quad (\pm 1)|\mathbf{T}| \cdot |\mathbf{w}_1| = 0,$$

ce qui entraîne, en supposant toujours

$$(28) \quad |\mathbf{w}_1| \neq 0$$

(car on parle toujours des deux axes instantanés de rotation, dont la direction est unique — cette fois — et bien déterminée), la valeur nulle de $\mathbf{T}$, la nécessité de la condition (23) étant, de cette façon, relevée.

Observation. Sans doute, une telle condition pour garder l'invariabilité de l'axe instantané de rotation, paraît être assez restrictive, lorsqu'on la considère être satisfaite d'une manière permanente, à la substitution d'un des repères cartésiens triorthogonaux par l'autre. C'est pour ça qu'on peut considérer seulement les moments, ou encore les intervalles (du temps), où cette condition est satisfaite, ce qui conduit à l'idée (à la notion) de l'invariabilité „instantanée“, ou à la notion de l'invariabilité „temporaire“ (ou „intermittente“) de l'axe instantané de rotation, ces notions pouvant avoir leur importance dans l'étude du mouvement d'un ensemble de corps rigides, pendant lequel les corps rigides qui constituent cet ensemble, peuvent se mouvoir instantanément, ou d'une manière temporaire, ou encore d'une manière permanente, par des translations, par rapport aux autres corps rigides qui constituent cet ensemble.

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ASUPRA INVARIANȚEI AXEI INSTANTANEE DE rotație A RIGIDULUI

II. Cazul a două repere carteziene triortogonale ce se mișcă exclusiv prin translație unul față de celălalt

(Rezumat)

Se stabilește condiția generală, necesară și suficientă, pentru ca două repere $X_1O_1Y_1Z_1$, $X_2O_2Y_2Z_2$, carteziene triortogonale, ce se mișcă exclusiv prin translație unul față de celălalt, să aibă aceeași axă instantanee de rotație în mișcarea acestor repere față de reperul fix, asemenea condiție lărgind pe aceea considerată în prima parte a lucrării [6].

Reperetele carteziene triortogonale la care ne referim sînt formate în exclusivitate din vectori unitari.

FUNDAMENTAL SOLUTIONS OF THE LINEAR MICROPOLAR ELASTOSTATICS EQUATIONS

BY

I. CRĂCIUN and ANA CRĂCIUN

1. Introduction. In this paper the calculation of fundamental solutions of the linear micropolar elastostatics equations of an isotropic and homogeneous infinite solid is given. For this purpose a new method presented in our previous paper [7] is used. The method was been suggested by the triple integral Fourier transforms and, it allows us to find directly fundamental solutions of the quoted equations without to use the Galerkin representation given by Șandru in [3] or the Nowacki's representations by potentials [1]. The same method has been used recently by Dragoș in [6] to obtain in other form these fundamental solutions. The other papers devoted to the same subject use or the Galerkin representation [4], [5], [8] etc., or the Nowacki's representations by elastic potentials [1], [2], etc.

Firstly, the application of the triple integral Fourier transform to the original field equations followed by the solution of the transformed equations is made. By using the inverse Fourier transform a general solution in a vectorial form in the infinite space of the differential equations governing the linear micropolar elastostatics theory of an isotropic and homogeneous infinite solid is given. The given solution is the same of that deduced by Nowacki ([1], p. 146) by potential representations. Then, in order to find fundamental solutions of the specified equations, the unitary concentrated loads of certain directions acting in the origin of the coordinate system $Ox_1x_2x_3$ are taken into consideration. Finally, using the linearity of the governing equations and the obtained fundamental solutions, the solutions of the micropolar elastostatics equations, when arbitrary concentrated loads are applied in a point of the space, are given. These solutions have been deduced by Șandru in [3] who used the Galerkin representation from [4].

2. General Solution of the Basic Equations. Following [1]–[4], the basic equations of the linear micropolar elastostatics theory of an isotropic and homogeneous solid, in terms of the displacement vector $\mathbf{u}$ and the rotation vector $\boldsymbol{\varphi}$, are

$$(2.1) \quad \begin{cases} (\mu + \alpha) \nabla^2 \mathbf{u} + (\lambda + \mu - \alpha) \nabla \nabla \cdot \mathbf{u} + 2\alpha \nabla \times \boldsymbol{\varphi} + \mathbf{X} = \mathbf{0}, \\ 2\alpha \nabla \times \mathbf{u} + [(\gamma + \varepsilon) \nabla^2 - 4\alpha] \boldsymbol{\varphi} + (\beta + \gamma - \varepsilon) \nabla \nabla \cdot \boldsymbol{\varphi} + \mathbf{Y} = \mathbf{0}, \end{cases}$$

where $\mathbf{X}$ is the body force vector; $\mathbf{Y}$ is the couple-body force vector; $\lambda, \mu, \alpha, \beta, \gamma, \varepsilon$ are the material constants and ∇ is the nabla operator: $\nabla f = f_{,i} \mathbf{e}_i$, when $f(x_1, x_2, x_3)$ is a scalar function, $\nabla \cdot \mathbf{v} = v_{i,i}$, when $\mathbf{v} = v_i \mathbf{e}_i$ is a vector function, $\nabla \times \mathbf{v} = \varepsilon_{ijk} v_{k,j} \mathbf{e}_i$ and $\nabla^2 = \nabla \cdot \nabla$ is the Laplace's operator. Here ε_{ijk} are the alternating symbols and x_i are the coordinates of a point x of the three-dimensional space E_3 referred to the rectangular coordinate system $Ox_1x_2x_3$, whose unitary vectors along of the axes are $\mathbf{e}_i$. The vectors from (2.1) are functions of the point x fulfilling certain regularity conditions in E_3 . The subscripts are understood to range over the integers 1, 2, 3. The usual summation convention over repeated indices is used. The subscripts preceded by comma denote partial differentiation with respect to corresponding coordinate. From the beginning we suppose that the solid occupies the entire space E_3 .

If we denote by $D(\partial/\partial x)$ the differential operator of the linear micropolar elastostatics theory for isotropic and homogeneous solids, then Eqs. (2.1) can be written as

$$(2.2) \quad D\left(\frac{\partial}{\partial x}\right)u + f = 0,$$

where $u = {}^t(\mathbf{u}, \varphi) = {}^t(u_1, u_2, u_3, \varphi_1, \varphi_2, \varphi_3)$, $f = {}^t(\mathbf{X}, \mathbf{Y})$ and t means the transpose operation.

To solve (2.1) in E_3 we introduce the triple integral Fourier transform $\tilde{f}(\xi)$ of every quantity $f(x)$ occurring in them, defined by

$$(2.3) \quad \tilde{f}(\xi) = (2\pi)^{-3/2} \int_{E_3} f(x) \exp(i\xi_k x_k) dx,$$

where $\xi(\xi_1, \xi_2, \xi_3)$, $i = \sqrt{-1}$ and $dx = dx_1 dx_2 dx_3$.

If we multiply both sides of each equation of the set (2.1) by $\exp(i\xi_k x_k)$, integrate throughout the entire space E_3 and make use of the results

$$(2.4) \quad (2\pi)^{-3/2} \int_{E_3} (f_{,j}, \nabla^2 f) \exp(i\xi_k x_k) dx = -(i\xi_j, \xi^2) \tilde{f},$$

where $\xi^2 = (\xi_k \xi_k)^{1/2}$, then we obtain the set

$$(2.5) \quad \begin{cases} (\mu + \alpha) \xi^2 \tilde{\mathbf{u}} + (\lambda + \mu - \alpha)(\xi \tilde{\mathbf{u}}) \xi + 2\alpha i \xi \times \tilde{\varphi} = \tilde{\mathbf{X}}, \\ 2\alpha i \xi \times \tilde{\mathbf{u}} + [(\gamma + \varepsilon) \xi^2 + 4\alpha] \tilde{\varphi} + (\beta + \gamma - \varepsilon)(\xi \tilde{\varphi}) \xi = \tilde{\mathbf{Y}}, \end{cases}$$

where $\xi = \xi_k \mathbf{e}_k$ is the position vector of the point $\xi(\xi_1, \xi_2, \xi_3)$ in the transformed space W_3 and $\mathbf{a} \cdot \mathbf{b}$ is the inner product of the $\mathbf{a}$ and $\mathbf{b}$ vectors.

By multiplying (2.5), scalar, with the ξ vector and by solving the system so derived into respect to the inner products $\xi \cdot \tilde{\mathbf{u}}$ and $\xi \cdot \tilde{\varphi}$, we obtain

$$(2.6) \quad \xi \cdot \tilde{\mathbf{u}} = \frac{(\xi \cdot \tilde{\mathbf{X}}) \xi}{(\lambda + 2\mu) \xi^2}, \quad \xi \cdot \tilde{\varphi} = \frac{(\xi \cdot \tilde{\mathbf{Y}}) \xi}{(\beta + 2\gamma) \xi^2 + 4\alpha}.$$

If we procede in (2.5) to the cross vector with the ξ vector and then we solve the obtained system into respect with $\xi \times \tilde{\mathbf{u}}$ and $\xi \times \tilde{\varphi}$, we get

$$\begin{aligned}
 (2.7) \quad \xi \times \tilde{\mathbf{u}} &= \frac{1}{\mu + \alpha} [\xi \times \tilde{\mathbf{X}} - 2i\alpha \xi \times (\xi \times \tilde{\varphi})], \\
 \xi \times \tilde{\varphi} &= \frac{1}{(\gamma + \varepsilon)\xi^2 + 4\alpha} [\xi \times \tilde{\mathbf{Y}} - 2i\alpha \xi \times (\xi \times \tilde{\mathbf{u}})],
 \end{aligned}$$

where $\xi \times (\xi \times \mathbf{a}) = (\xi \cdot \mathbf{a})\xi - \xi^2 \mathbf{a}$, is the double cross vector.

By introducing (2.6) and (2.7) in (2.5), we obtain

$$(2.8) \quad \left\{ \begin{aligned} \tilde{\mathbf{u}} &= \frac{\xi^2 + 2p}{(\mu + \alpha)\xi^2(\xi^2 + \sigma_4^2)} \tilde{\mathbf{X}} - \frac{ip}{(\mu + \alpha)\xi^2(\xi^2 + \sigma_4^2)} \xi \times \tilde{\mathbf{Y}} - \\ &\quad - \frac{(\lambda + \mu - \alpha)\xi^2 + 2p(\lambda + \mu)}{(\mu + \alpha)(\lambda + 2\mu)\xi^4(\xi^2 + \sigma_4^2)} (\xi \cdot \tilde{\mathbf{X}})\xi, \\ \tilde{\varphi} &= \frac{1}{(\gamma + \varepsilon)(\xi^2 + \sigma_4^2)} \tilde{\mathbf{Y}} - \frac{is}{(\gamma + \varepsilon)\xi^2(\xi^2 + \sigma_4^2)} \xi \times \tilde{\mathbf{X}} - \\ &\quad - \frac{(\beta + \gamma - \varepsilon)\xi^2 + 2\alpha s}{(\beta + 2\gamma)(\gamma + \varepsilon)\xi^2(\xi^2 + \sigma_3^2)(\xi^2 + \sigma_4^2)} (\xi \cdot \tilde{\mathbf{Y}})\xi, \end{aligned} \right.$$

where the following notations had been made

$$\sigma_3^2 = \frac{4\alpha}{\beta + 2\gamma}, \quad \sigma_4^2 = \frac{4\alpha\mu}{(\mu + \alpha)(\gamma + \varepsilon)}, \quad p = \frac{2\alpha}{\gamma + \varepsilon}, \quad s = \frac{2\alpha}{\mu + \alpha}.$$

In this way we have obtained, in a simple manner, the solution in W_3 of the transformed system (2.5). This solution can be written as

$$(2.9) \quad \left\{ \begin{aligned} \tilde{\mathbf{u}} &= \left[\frac{1}{\mu\xi^2} - \frac{\alpha}{\mu(\mu + \alpha)} \frac{1}{\xi^2 + \sigma_4^2} \right] \tilde{\mathbf{X}} - \frac{i}{2\mu} \left(\frac{1}{\xi^2} - \frac{1}{\xi^2 + \sigma_4^2} \right) \xi \times \tilde{\mathbf{Y}} + \\ &\quad + \frac{1}{4\mu} \left[\frac{\gamma + \varepsilon}{\mu} \left(\frac{1}{\xi^2} - \frac{1}{\xi^2 + \sigma_4^2} \right) - \frac{4(\lambda + \mu)}{\lambda + 2\mu} \frac{1}{\xi^4} \right] (\xi \cdot \tilde{\mathbf{X}})\xi, \\ \tilde{\varphi} &= \frac{1}{\gamma + \varepsilon} \frac{1}{\xi^2 + \sigma_4^2} \tilde{\mathbf{Y}} - \frac{i}{2\mu} \left(\frac{1}{\xi^2} - \frac{1}{\xi^2 + \sigma_4^2} \right) \xi \times \tilde{\mathbf{X}} + \\ &\quad + \frac{1}{4} \left[\left(\frac{1}{\mu} + \frac{1}{\alpha} \right) \frac{1}{\xi^2 + \sigma_4^2} - \frac{1}{\mu\xi^2} - \frac{1}{\alpha\xi^2 + \sigma_4^2} \right] (\xi \cdot \tilde{\mathbf{Y}})\xi. \end{aligned} \right.$$

To obtain the corresponding expressions both of the displacement vector and the rotation vector, we make use of the Fourier's integral theorem for three-dimensional transforms which states that if $\tilde{f}(\xi)$ is defined in terms of $f(x)$ by (2.3), then

$$(2.10) \quad f(x) = (2\pi)^{-3/2} \int_{W_3} \tilde{f}(\xi) \exp(-i\xi_k x_k) d\xi,$$

where $d\xi = d\xi_1 d\xi_2 d\xi_3$.

On inverting (2.9) by the rule (2.10), we find that

$$(2.11) \quad \begin{cases} \mathbf{u}(x) = (2\pi)^{-3/2} \int_{W_3} \tilde{\mathbf{u}}(\xi) \exp(-i\xi_k x_k) d\xi, \\ \varphi(x) = (2\pi)^{-3/2} \int_{W_3} \tilde{\varphi}(\xi) \exp(-i\xi_k x_k) d\xi, \end{cases}$$

who is the general solution in E_3 of the original system (2.1), the same with that derived by Nowacki ([1], p. 146), by an other method.

3. Fundamental Solution of Eqs. (2.1). As it is well-known, we call as fundamental solutions of the linear micropolar elastostatics equations of an isotropic and homogeneous body the solutions in E_3 of the set (2.1) corresponding to concentrated loads applied in an arbitrary point of the space ([4], p. 164).

Let us consider, at the begining, that in the origin of the coordinate system acts a unitary concentrated force of $\mathbf{e}_k$ -direction, that is

$$(3.1) \quad \mathbf{X}(x) = \delta(x) \delta_k, \quad \mathbf{Y}(x) = 0, \quad x \in E_3,$$

where $\delta(x)$ is the Dirac's distribution and $\delta_k = \delta_{kj} \mathbf{e}_j = \mathbf{e}_k$. Taking into account, that

$$(3.2) \quad \int_{W_3} \delta(x) \exp(-i\xi_k x_j) dx = 1,$$

from (2.3) and (3.1), (3.2), we obtain

$$(3.3) \quad \tilde{\mathbf{X}}(\xi) = (2\pi)^{-3/2} \delta_k, \quad \mathbf{Y} = 0, \quad \xi \in W_3.$$

By introducing (3.3) in (2.9) and making use of the results ([1], p. 121)

$$(3.4) \quad \begin{cases} \int_{W_3} \left(\frac{1}{\xi^2 + k^2}, \frac{1}{\xi_i} \right) \exp(-i\xi_s x_s) d\xi = \pi^2 \left(\frac{2}{R} \exp(-kR), -R \right), \\ \int_{W_3} \frac{1}{\xi^2 + k^2} \xi_j \exp(-i\xi_s x_s) d\xi = 2\pi^2 \left(\frac{1}{R} \exp(-kR) \right)_{,j}, \\ \int_{W_3} \left(\frac{1}{\xi^2 + k^2}, \frac{1}{\xi_i} \right) \xi_k \xi_j \exp(-i\xi_s x_s) d\xi = \pi^2 \left(-\frac{2}{R} \exp(-kR) \right)_{,kj}, \end{cases}$$

where $R = (x_j x_j)^{1/2}$, we obtain for $\mathbf{u}_k(x)$ and $\varphi_k(x)$ the following expressions

$$(3.5) \quad \begin{cases} \mathbf{u}_k = \frac{1}{4\pi\mu} \left(\frac{1}{R} - \frac{\alpha}{\mu + \alpha} \frac{\exp(-\sigma_4 R)}{R} \right) \delta_k - \frac{1}{8\pi\mu} \frac{\partial^2}{\partial x_i \partial x_k} \left(\frac{\lambda + \mu}{\lambda + 2\mu} R + \right. \\ \quad \left. + \frac{\gamma + \epsilon}{2\mu} \frac{1 - \exp(-\sigma_4 R)}{R} \right) \mathbf{e}_i, \\ \varphi_k = \frac{1}{8\pi\mu} \epsilon_{kij} \frac{\partial}{\partial x_j} \left(\frac{1 - \exp(-\sigma_4 R)}{R} \right) \mathbf{e}_i, \end{cases}$$

which are the same as in ([1], p. 147) and 1/2 from the similar result given in ([2], p. 73.)

If in (3.5) we calculate the mentioned derivatives, then, after a simple calculation we get

$$(3.6) \quad \begin{cases} \mathbf{u}_k = \frac{1}{16\pi\mu(1-\nu)} \left[(3-4\nu) \frac{\delta_k}{R} + \frac{(\mathbf{R}\delta_k)\mathbf{R}}{R^3} \right] + \frac{l^2}{4\pi\mu} \left\{ [1 - (1+\sigma_4 R)\exp(-\sigma_4 R)] \times \right. \\ \left. \times \left(\frac{\delta_k}{R^3} - 3 \frac{(\mathbf{R}\delta_k)\mathbf{R}}{R^5} \right) - \sigma_4^2 \exp(-\sigma_4 R) \frac{\mathbf{R} \times (\delta_k \times \mathbf{R})}{R^3} \right\}, \\ \varphi_k = \frac{1}{8\pi\mu} [1 - (1+\sigma_4 R)\exp(-\sigma_4 R)] \frac{\delta_k \times \mathbf{R}}{R^3}, \quad \mathbf{R} = x_i \mathbf{e}_i, \end{cases}$$

where $\nu = \lambda/(2(\lambda + \mu))$, $l^2 = (\gamma + \varepsilon)/(4\mu)$. The expressions of the solution $(\mathbf{u}_k, \varphi_k)$ given in (3.6) are the same derived by Șandru in ([4], p. 174,) by using the Galerkin representation from [3]. Obviously, we have

$$D \left(\frac{\partial}{\partial x} \right)^t (\mathbf{u}_k, \varphi_k) + {}^t(\delta(x)\delta_k, 0) = 0, \quad x \in E_3.$$

If the unitary concentrated force is applied in the point $y(y_1, y_2, y_3)$, and acts on the direction of Ox_k axis, then the corresponding solution of (2.2), where $f = {}^t(\delta(x-y)\delta_k, 0)$, is obtained from (3.6) by passing $\mathbf{R}$ and R into $\mathbf{r} = (x_i - y_i)\mathbf{e}_i$ and $r = [(x_j - y_j)(x_j - y_j)]^{1/2}$, respectively.

The knowledge of the solution (3.6) allow us to write the solution corresponding to the action of every concentrated force placed in the origin of the coordinate system. Really, on the base of the linearity of (2.2) it results that the solution of

$$D \left(\frac{\partial}{\partial x} \right) u + {}^t(\delta(x)\mathbf{Q}, 0) = 0, \quad x \in E_3,$$

is $u = {}^t(\mathbf{u}, \varphi)$, given by $\mathbf{u} = Q_k \mathbf{u}_k$, $\varphi = Q_k \varphi_k$, where $\mathbf{Q} = Q_k \mathbf{e}_k$ and $\mathbf{u}_k, \varphi_k$ are the same as in (3.6).

If we suppose that a unitary concentrated couple of the Ox_k direction acts in the origin of the coordinate system, then we have

$$(3.7) \quad \mathbf{X}(x) = 0, \quad \mathbf{Y}(x) = \delta(x)\delta_k, \quad x \in E_3.$$

From (3.7) and (3.2), we obtain

$$(3.8) \quad \tilde{\mathbf{X}}(\xi) = 0, \quad \tilde{\mathbf{Y}}(\xi) = (2\pi)^{-3/2} \delta_k, \quad \xi \in W_3.$$

Introduction of (3.8) into (2.11), yields

$$(3.9) \quad \begin{cases} \mathbf{U}_k = \frac{1}{8\pi\mu} [1 - (1+\sigma_4 R)\exp(-\sigma_4 R)] \frac{\delta_k \times \mathbf{R}}{R^3}, \\ \Phi_k = \frac{1}{16\pi\alpha} \left[\left(\frac{\delta_k}{R^3} - 3 \frac{(\mathbf{R}\delta_k)\mathbf{R}}{R^5} \right) [1 - (1+\sigma_3 R)\exp(-\sigma_3 R)] + \right. \end{cases}$$

$$(3.9) \quad \left[\begin{aligned} & + \sigma_3^2 \exp(-\sigma_3 R) \frac{(R\delta_k)R}{R^3} - \frac{\mu + \alpha}{16\pi\mu\alpha} \left[1 - (1 + \sigma_4 R) \times \right. \\ & \left. \times \exp(-\sigma_4 R) \right] \left(\frac{\delta_k}{R^3} - 3 \frac{(R\delta_k)R}{R^5} \right) + \sigma_4^2 \exp(-\sigma_4 R) \left(\frac{(R\delta_k)R}{R^3} - \frac{\delta_k}{R} \right) \right], \end{aligned} \right.$$

which is the solution of the equation

$$D\left(\frac{\partial}{\partial x}\right)u + {}^t(0, \delta(x)\delta_k) = 0, \quad x \in E_3,$$

the same as the solution derived by Șandru in ([4], p. 176.) who has been started from the Galerkin representation given in ([4], p. 156).

If the unitary concentrated couple acting on the Ox_k direction is placed in the point $y(y_1, y_2, y_3)$ then the corresponding solution may be obtained from (3.9) by passing R and R into r and r , respectively.

From the same reason as in above we get that the solution of

$$D\left(\frac{\partial}{\partial x}\right)u + {}^t(0, \delta(x)M) = 0, \quad x \in E_3,$$

is $u = {}^t(U, \Phi)$ given by $U = M_k U_k$, $\Phi = M_k \Phi_k$, where $M = M_k e_k$ is a constant vector.

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SOLUȚII FUNDAMENTALE ALE ECUAȚIILOR ELASTOSTATICII MICROPOLARE LINIARE

(Rezumat)

Se determină soluții fundamentale pentru ecuațiile elastostaticii micropolare ale unui solid infinit izotrop și omogen. În acest scop se folosește o metodă prezentată într-un articol precedent. Metoda a fost sugerată de către transformatele integrale triple Fourier și permite găsirea directă a soluțiilor fundamentale fără folosirea reprezentării Galerkin [3] sau a reprezentărilor prin potențiali [1].

SYSTEMATIC METHODS OF APPROXIMATION IN NONLINEAR THEORY OF COSSERAT MEDIA (III)

BY

I. NISTOR

0. Introduction. In this paper we give a perturbation scheme of Signorini's type for initial-value problem of traction and couple in finite elastodynamics of micropolar media. We shall present an interpretation of the expansion following Rivlin and T o p a k o g l u [1].

1. Basic Equations. Throughout this paper a rectangular coordinate system is employed. The motion of a Cosserat medium is described [2] by the functions

$$(1.1) \quad y_i = y_i(x_j, t), \quad R_{ij} = R_{ij}(x_k, t),$$

where x_i and y_j are the coordinates of the place occupied by the body point in the reference configuration and the current configuration, respectively; R_{ij} is a proper orthogonal tensor that characterizes the rotation undergone by the body point.

If we denote by φ_i the components of microrotation vector, then R_{ij} can be represented in the form [3]

$$(1.2) \quad R_{ij} = \frac{1}{1 + \frac{1}{4} \varphi_h \varphi_h} \left[\left(1 - \frac{1}{4} \varphi_k \varphi_k \right) \delta_{ij} + \frac{1}{2} \varphi_i \varphi_j + \epsilon_{jlp} \varphi_p \right].$$

In this paper the following strain tensors are used :

$$(1.3) \quad e_{ij} = y_{k,i} R_{kj} - \delta_{ij}, \quad \gamma_{ij} = \frac{1}{2} \epsilon_{jmn} R_{kn} R_{km,i},$$

where the comma denotes partial derivation with respect to the variables x_i . The components v_i of the angular velocity vector are given by

$$(1.4) \quad v_i = \frac{1}{2} \epsilon_{ijl} R_{jp} \dot{R}_{lp},$$

where a superposed dot indicates material differentiation.

The equations of motion of Cosserat media are [2]

$$(1.5) \quad \begin{aligned} s_{ij,i} + \rho F_j &= \rho \ddot{u}_j, \\ \mu_{ijk,i} + \epsilon_{jlk} y_{i,p} s_{pk} + \rho G_j &= \rho \dot{\sigma}_j, \end{aligned}$$

where

$$(1.6) \quad s_{ij} = \frac{\partial W}{\partial e_{ik}} R_{jk}, \quad \mu_{ij} = \frac{\partial W}{\partial \gamma_{ik}} R_{jk}, \quad W = W(e_{ij}, \gamma_{pq}),$$

$$(1.7) \quad \sigma_i = J_{ik} v_k,$$

$$(1.8) \quad J_{ik} = R_{ip} R_{kq} I_{pq}, \quad I_{pq} = 0,$$

$$(1.9) \quad G_i = e_{ijk} R_{jp} B_{pk}.$$

In these relations we have used the following notation: s_{ij} and μ_{ij} — components of stress and couple stress tensors measured per unit area of reference configuration; F_i — components of body force vector; B_{jk} — components of body generalized force vectors; ρ — density of the body in reference configuration; I_{pq} — components of the material tensor of inertia density; σ_i — components of the microspin vector; W — strain energy per unit volume of reference configuration.

In the case of prescribed values of surface forces f_i and generalized surface b_{jk} , the boundary conditions are written as

$$(1.10) \quad s_{ij} N_i = f_j, \quad \mu_{ij} N_i = g_j,$$

where N_i are the components of the outward unit normal to the surface and g_i are the components of surface couples given by

$$(1.11) \quad g_i = e_{ijk} R_{jp} b_{pk}.$$

2. Expansions of Signorini's Type. For simplicity we shall use the initial configuration as the reference configuration. We assume that the reference configuration is a natural state and that the strain energy function is an analytic function on e_{ij} and γ_{pq} . We then have an expansion

$$(2.1) \quad W = \sum_{p=1}^{\infty} \widetilde{W}^{(p)},$$

where $\widetilde{W}^{(p)}$ is a homogeneous polynomial of degree p in e_{ij} and γ_{pq} . The relations (1.2) and (1.3) may be written in the form

$$(2.2) \quad R_{ij} = \sum_{p=0}^{\infty} \widetilde{R}_{ij}^{(p)}, \quad e_{ij} = \sum_{p=1}^{\infty} \widetilde{e}_{ij}^{(p)}, \quad \gamma_{ij} = \sum_{p=1}^{\infty} \widetilde{\gamma}_{ij}^{(p)},$$

where

$$(2.3) \quad \widetilde{R}_{ij}^{(0)} = \delta_{ij}, \quad \widetilde{e}_{ij}^{(1)} = u_{j,i} + e_{jlk} \varphi_k, \quad \widetilde{\gamma}_{ij}^{(1)} = \varphi_{j,i}.$$

Here $\widetilde{R}_{ij}^{(p)}$, $\widetilde{e}_{ij}^{(p)}$, $\widetilde{\gamma}_{ij}^{(p)}$ ($p \geq 1$) are homogeneous polynomials of degree p in $u_{j,i}$, φ_i and $\varphi_{i,j}$.

From (1.6), (2.1) and (2.2) we see that the stress tensor s_{ij} and couple stress tensor μ_{ij} have the series expansions

$$(2.4) \quad s_{ij} = \sum_{p=1}^{\infty} \widetilde{s}_{ij}^{(p)}, \quad \mu_{ij} = \sum_{p=1}^{\infty} \widetilde{\mu}_{ij}^{(p)},$$

where $\tilde{s}_{ij}^{(p)}$ and $\tilde{\mu}_{ij}^{(p)}$ are polynomials of degree p in $u_{i,j}$, φ_i , $\varphi_{i,j}$ and where

$$(2.5) \quad \begin{aligned} \tilde{s}_{ij}^{(1)}(u_{h,r}, \varphi_t, \varphi_{n,m}) &= \frac{\partial^2 W}{\partial e_{ij} \partial e_{qp}} (u_{p,q} + e_{pqk} \varphi_k) + \frac{\partial^2 W}{\partial e_{ij} \partial \gamma_{pq}} \varphi_{q,p}, \\ \tilde{\mu}_{ij}^{(1)}(u_{h,r}, \varphi_t, \varphi_{n,m}) &= \frac{\partial^2 W}{\partial \gamma_{ij} \partial e_{pq}} (u_{q,p} + e_{qp k} \varphi_k) + \frac{\partial^2 W}{\partial \gamma_{ij} \partial \gamma_{pq}} \gamma_{q,p}. \end{aligned}$$

Eq. (1.7) differentiated with respect to time becomes

$$(2.6) \quad \dot{\sigma}_i = \sum_{p=1}^{\infty} \tilde{\sigma}_i^{(p)},$$

where $\tilde{\sigma}_i^{(p)}$ is homogeneous polynomial of degree p in φ_i , $\dot{\varphi}_j$, $\ddot{\varphi}_{i,j}$ and where

$$(2.7) \quad \tilde{\sigma}_j^{(1)} = I_{jk} \ddot{\varphi}_k.$$

We assume that the body forces F_i , body generalized forces $B_{i,j}$, surface tractions f_i and surface generalized forces $b_{i,j}$ depend analytically on a parameter ε and they vanish when $\varepsilon=0$. We then have series expansions

$$(2.8) \quad F_i = \sum_{p=1}^{\infty} \varepsilon^p F_i^{(p)}, \quad B_{i,j} = \sum_{p=1}^{\infty} \varepsilon^p B_{i,j}^{(p)}, \quad f_i = \sum_{p=1}^{\infty} \varepsilon^p f_i^{(p)}, \quad b_{i,j} = \sum_{p=1}^{\infty} \varepsilon^p b_{i,j}^{(p)}.$$

We assume that there exists a solution (u_i, φ_j) to the boundary-value problem defined by the loads (2.8) which depends analytically on ε and vanishes when $\varepsilon=0$. In other words we assume the existence of a solution of the form

$$(2.9) \quad u_i = \sum_{p=1}^{\infty} \varepsilon^p u_i^{(p)}, \quad \varphi_j = \sum_{p=1}^{\infty} \varepsilon^p \varphi_j^{(p)}.$$

Substitution of (2.9) into (2.2)₁ and (2.4) shows that the microrotation tensor R_{ij} , stress tensor s_{ij} and couple stress tensor μ_{ij} have series expansions

$$(2.10) \quad R_{ij} = \delta_{ij} + \sum_{p=1}^{\infty} \varepsilon^p R_{ij}^{(p)}, \quad s_{ij} = \sum_{p=1}^{\infty} \varepsilon^p s_{ij}^{(p)}, \quad \mu_{ij} = \sum_{p=1}^{\infty} \varepsilon^p \mu_{ij}^{(p)}.$$

Introducing the notation

$$\tilde{s}_{ij}^{(1)}(u_{h,r}^{(p)}, \varphi_t^{(p)}, \varphi_{n,m}^{(p)}) = \tilde{s}_{ij}^{(1)}(z_k^{(p)}), \quad \tilde{\mu}_{ij}^{(1)}(u_{h,r}^{(p)}, \varphi_t^{(p)}, \varphi_{n,m}^{(p)}) = \tilde{\mu}_{ij}^{(1)}(z_k^{(p)}),$$

we can represent the terms $s_{ij}^{(p)}$ and $\mu_{ij}^{(p)}$ in the form

$$(2.11) \quad s_{ij}^{(p)} = \tilde{s}_{ij}^{(1)}(z_k^{(p)}) + \hat{s}_{ij}^{(p)}, \quad \mu_{ij}^{(p)} = \tilde{\mu}_{ij}^{(1)}(z_k^{(p)}) + \hat{\mu}_{ij}^{(p)},$$

where the polynomial functions $\hat{s}_{ij}^{(p)}$, $\hat{\mu}_{ij}^{(p)}$ depend on $u_{i,j}^{(1)}$, $u_{i,j}^{(2)}, \dots, u_{i,j}^{(p-1)}$, $\varphi_i^{(1)}$, $\varphi_i^{(2)}, \dots, \varphi_i^{(p-1)}$, $\varphi_{i,j}^{(1)}$, $\varphi_{i,j}^{(2)}, \dots, \varphi_{i,j}^{(p-1)}$ and in particular $\hat{s}_{ij}^{(1)} = 0$, $\hat{\mu}_{ij}^{(1)} = 0$.

Using (2.10)₁ we obtain that $\dot{\sigma}_j$ has a series expansion

$$(2.12) \quad \dot{\sigma}_j = \sum_{p=1}^{\infty} \varepsilon^p \sigma_j^{(p)}.$$

The terms $\sigma_j^{(p)}$ of expansion (2.12) may be represented in the form

$$(2.13) \quad \sigma_j^{(p)} = I_{jk} \varphi_k^{(p)} + \hat{\sigma}_j^{(p)},$$

where $\hat{\sigma}_j^{(p)}$ depends on $\varphi_i^{(1)}, \varphi_i^{(2)}, \dots, \varphi_i^{(p-1)}, \dot{\varphi}_j^{(1)}, \dot{\varphi}_j^{(2)}, \dots, \dot{\varphi}_j^{(p-1)}, \ddot{\varphi}_i^{(1)}, \ddot{\varphi}_i^{(2)}, \dots, \ddot{\varphi}_i^{(p-1)}$. In particular $\hat{\sigma}_i^{(1)} = 0$.

Substituting (2.8)_{2,4} into (1.9) and (1.11) we obtain

$$(2.14) \quad G_j = \sum_{p=1}^{\infty} \varepsilon^p G_j^{(p)}, \quad g_i = \sum_{p=1}^{\infty} \varepsilon^p g_i^{(p)},$$

where

$$(2.15) \quad G_j^{(p)} = \sum_{\alpha=0}^{p-1} e_{jik} R_{ih}^{(\alpha)} B_{hk}^{(p-\alpha)}, \quad g_j^{(p)} = \sum_{\alpha=0}^{p-1} e_{jik} R_{ih}^{(\alpha)} b_{hk}^{(p-\alpha)}.$$

Substituting (2.8)_{1,3}, (2.9), (2.10), (2.12) and (2.14) into (1.5) and (1.10) and then equating like powers of ε yields the following successive systems of differential equations and associated boundary conditions:

$$(2.16) \quad \begin{aligned} s_{ij,i}^{(k)} + \rho F_j^{(k)} &= \rho \tilde{u}_j^{(k)}, \\ \mu_{ij,i}^{(k)} + e_{jit} s_{ih}^{(k)} + \sum_{\alpha=1}^{k-1} e_{jit} u_{ih}^{(\alpha)} s_{hp}^{(k-\alpha)} + \rho G_j^{(k)} &= \rho \sigma_j^{(k)}, \\ s_{ij}^{(k)} N_i &= f_j^{(k)}, \quad \mu_{ij}^{(k)} N_j = g_j^{(k)}. \end{aligned}$$

When (2.11) and (2.13) is used, these systems become

$$(2.17) \quad \begin{aligned} \tilde{s}_{ij,i}^{(1)}(z_h^{(n)}) + \rho \tilde{F}_j^{(n)} &= \rho \tilde{u}_j^{(n)}, \\ \tilde{\mu}_{ij,i}^{(1)}(z_h^{(n)}) + e_{jit} \tilde{s}_{ih}^{(1)}(z_h^{(n)}) + \rho \tilde{G}_j^{(n)} &= \rho I_{jk} \ddot{\varphi}_k^{(n)}, \\ \tilde{s}_{ij}^{(1)}(z_h^{(n)}) N_i &= \tilde{f}_j^{(n)}, \quad \tilde{\mu}_{ij}^{(1)}(z_h^{(n)}) N_i = \tilde{g}_j^{(n)}, \end{aligned}$$

where

$$(2.18) \quad \begin{aligned} \rho \tilde{F}_j^{(n)} &= \rho F_j^{(n)} + \hat{s}_{ij,i}^{(n)}, \\ \rho \tilde{G}_j^{(n)} &= \rho G_j^{(n)} + \hat{\mu}_{ij,i}^{(n)} + e_{jit} \left(\hat{s}_{ik}^{(n)} + \sum_{\alpha=1}^{n-1} u_{ih}^{(\alpha)} s_{hk}^{(n-\alpha)} \right) - \rho \hat{\sigma}_j^{(n)}, \\ \tilde{f}_j^{(n)} &= f_j^{(n)} - \hat{s}_{ij}^{(n)} N_i, \quad \tilde{g}_j^{(n)} = g_j^{(n)} - \hat{\mu}_{ij}^{(n)} N_i. \end{aligned}$$

Since $\hat{s}_{ij}^{(1)} = \hat{\mu}_{ij}^{(1)} = 0$, $\hat{\sigma}_i^{(1)} = 0$, when $n=1$ we have $\tilde{F}_j^{(1)} = F_j^{(1)}$, $\tilde{G}_j^{(1)} = G_j^{(1)}$, $\tilde{f}_j^{(1)} = f_j^{(1)}$, $\tilde{g}_j^{(1)} = g_j^{(1)}$.

Hence the equations (2.17) are those of the line theory of micropolar elasticity, where $F_j^{(1)}$, $G_j^{(1)}$, $f_i^{(1)}$, $g_i^{(1)}$ are the assigned external loads. Let it be supposed that the fields $(u_i^{(1)}, \varphi_j^{(1)})$, $(u_i^{(2)}, \varphi_j^{(2)})$, ..., $(u_i^{(m-1)}, \varphi_j^{(m-1)})$ satisfying (2.17) for $n=1, 2, \dots, m-1$ have been determined. Then for $n=m$

(2.17) has the form of the equations defining the second boundary-value problem of linear micropolar elasticity for the same material and the same boundary though subject to assigned external load which depends in an explicitly known way upon the previously determined fields $(u_i^{(1)}, \varphi_j^{(1)})$, $(u_i^{(2)}, \varphi_j^{(2)})$, ..., $(u_i^{(m-1)}, \varphi_j^{(m-1)})$. Thus the calculation of n -terms in the expansion (2.9) is reduced formally, to the solution of n initial-value and boundary-value problems of traction and couple in the infinitesimal theory of micropolar elasticity for the same body.

3. Physical Interpretation of the Method. Using the method outlined in [1] we present an interpretation of the expansions of Sect. 2.

Let us denote the partial sums of the expansions (2.4), (2.6), (2.9), (2.8)_{1,3}, (2.10)_{2,3}, (2.12) and (2.14) by $\tilde{s}_{ij}^n$, $\tilde{\mu}_{ij}^n$, F_i^n , f_j^n , $\tilde{\sigma}_i^n$, u_i^n , φ_j^n , s_{ji}^n , μ_{ij}^n , σ_i^n , G_j^n , g_i^n , respectively. For example

$$(3.1) \quad s_{ij}^n = \sum_{p=1}^n \varepsilon^p s_{ij}^{(p)}, \quad \mu_{ij}^n = \sum_{p=1}^n \varepsilon^p \mu_{ij}^{(p)}.$$

It follows from (2.16) that

$$(3.2) \quad \begin{aligned} \tilde{s}_{ij,i}^n + \rho F_j^n &= \rho \tilde{u}_j^n, \\ \mu_{ij,i}^n + e_{jkl} \left(\tilde{s}_{ik}^n + \sum_{p=1}^{n-1} \varepsilon^p u_{i,h}^{(p)} s_{hk}^{n-p} \right) + \rho G_j^n &= \rho \sigma_j^n, \\ \tilde{s}_{ij}^n N_i &= f_j^n, \quad \mu_{ij}^n N_i = g_j^n. \end{aligned}$$

The notation

$$\begin{aligned} \tilde{s}_{kj}^n(u_{h,i}^m, \varphi_r^m, \varphi_{p,q}^m) &= \tilde{s}_{kj}^n(z_i^m), \quad \tilde{\mu}_{jk}^n(u_{h,i}^m, \varphi_r^m, \varphi_{p,q}^m) = \tilde{\mu}_{jk}^n(z_i^m), \\ \tilde{\sigma}_q^n(\varphi_i^m, \varphi_j^m, \varphi_k^m) &= \tilde{\sigma}_q^n(h_i^m), \end{aligned}$$

are used in the following.

It is easily verified that

$$(3.3) \quad \tilde{s}_{jh}^n(z_i^n) = s_{jh}^n + O(\varepsilon^{n+1}), \quad \tilde{\mu}_{ij}^n(z_k^n) = \mu_{ij}^n + O(\varepsilon^{n+1}), \quad \tilde{\sigma}_q^n(h_i^n) = \sigma_q^n + O(\varepsilon^{n+1}),$$

where $O(\varepsilon^{n+1})$, indicates terms of order ε^{n+1} and higher. Combining (3.2) and (3.3) we obtain

$$(3.4) \quad \begin{aligned} \tilde{s}_{ij,i}^n(z_k^n) + \rho F_j^n &= \rho \tilde{u}_j^n + O(\varepsilon^{n+1}), \\ \mu_{ij,i}^n(z_k^n) + e_{jlp} \tilde{s}_{ip}^n(z_k^n) + u_{i,q}^{n-1} \tilde{s}_{qp}^{n-1}(z_k^{n-1}) + \rho G_j^n &= \rho \tilde{\sigma}_j^n(h_i^n) + O(\varepsilon^{n+1}), \\ \tilde{s}_{ij}^n(z_k^n) N_i &= f_j^n + O(\varepsilon^{n+1}), \quad \tilde{\mu}_{ij}^n(z_q^n) N_i = g_j^n + O(\varepsilon^{n+1}). \end{aligned}$$

We shall say we have solved the equations of n -th order micropolar elasticity, if we have found a polynomial solution

$$u_i^n = \sum_{p=1}^n \varepsilon^p u_i^{(p)}, \quad \varphi_j^n = \sum_{p=1}^n \varepsilon^p \varphi_j^{(p)},$$

of the system (3.4). The partial sum (u_i^n, φ_j^n) of Signorini's expansion (2.9) gives such a solution and every solution is of this form.

Using the method given in ([4], Sect. 65) we obtain

$$\begin{aligned}
 \tilde{s}_{ij}^n(z_k^{n-1}) &= s_{ij}^{n-1} + \varepsilon^n \hat{s}_{ij}^n + O(\varepsilon^{n+1}), \\
 \tilde{\mu}_{ij}^n(z_k^{n-1}) &= \mu_{ij}^{n-1} + \varepsilon^n \hat{\mu}_{ij}^n + O(\varepsilon^{n+1}), \\
 (3.5) \quad \tilde{\sigma}_j(h_i^{n-1}) &= \sigma_j^{n-1} + \varepsilon^n \hat{\sigma}_j^n + O(\varepsilon^{n+1}), \\
 u_{i,j}^{n-1} s_{jk}^{n-1}(z_q^{n-1}) &= \sum_{\alpha=1}^{n-2} \varepsilon^\alpha u_{i,p}^{(\alpha)} s_{pk}^{n-1-\alpha} + \sum_{\alpha=1}^{n-1} \varepsilon^n u_{i,p}^{(\alpha)} s_{pk}^{(n-\alpha)} + O(\varepsilon^{n+1}).
 \end{aligned}$$

Writing (3.2) with n replaced by $n-1$ and substituting (3.5) into the result we find

$$\begin{aligned}
 \tilde{s}_{ij,i}^n(z_k^{n-1}) + \rho(F_j^n - \varepsilon^n \tilde{F}_j^{(n)}) &= \rho \tilde{u}_j^{n-1} + O(\varepsilon^{n+1}), \\
 \mu_{ijk}^n(z_k^{n-1}) + e_{ijk} \tilde{s}_{ik}^n(z_p^{n-1}) + u_{i,q}^{n-1} s_{qk}^{n-1}(z_p^{n-1}) + \\
 (3.6) \quad &+ \rho(G_j^n - \varepsilon^n \tilde{G}_j^{(n)}) = \rho \tilde{\sigma}_j^n(h_i^{n-1}) + O(\varepsilon^{n+1}), \\
 \tilde{s}_{ij}^n(z_q^{n-1}) N_i &= f_j^n - \varepsilon^n \tilde{f}_j^{(n)} + O(\varepsilon^{n+1}), \\
 \tilde{\mu}_{ij}^n(z_q^{n-1}) N_i &= g_j^n - \varepsilon^n \tilde{g}_j^{(n)} + O(\varepsilon^{n+1}),
 \end{aligned}$$

where $\tilde{F}_i^{(n)}, \tilde{G}_i^{(n)}, \tilde{f}_i^{(n)}, \tilde{g}_i^{(n)}$ are given by (2.18).

Suppose that we have a solution $(u_i^{n-1}, \varphi_j^{n-1})$ of the equations of $n-1$ -order micropolar elasticity in the sense defined above. Equations (3.6) shows that $-\varepsilon^n \tilde{F}_i^{(n)}, -\varepsilon^n \tilde{G}_j^{(n)}$ are the body loads and $-\varepsilon^n \tilde{f}_i^{(n)}, -\varepsilon^n \tilde{g}_j^{(n)}$ surface loads that would be required, in addition to the assigned loads $F_j^n, G_i^n, f_j^n, g_k^n$, in order to maintain the displacement u_i^{n-1} and microrotation φ_j^{n-1} in equilibrium according to the n -order theory of micropolar elasticity. Eqs. (2.17) show that $(\varepsilon^n u_i^{(n)}, \varepsilon^n \varphi_j^{(n)})$ is a solution in the infinitesimal theory corresponding to the loads

$$(\varepsilon^n \tilde{F}_i^{(n)}, \varepsilon^n \tilde{G}_j^{(n)}, \varepsilon^n \tilde{f}_i^{(n)}, \varepsilon^n \tilde{g}_k^{(n)}).$$

We now prove that a solution of n -th-order theory is given by

$$(3.7) \quad u_i^n = u_i^{n-1} + \varepsilon^n u_i^{(n)}, \quad \varphi_j^n = \varphi_j^{n-1} + \varepsilon^n \varphi_j^{(n)}.$$

From (2.11), (3.3) and (3.5) we obtain

$$\begin{aligned}
 (3.8) \quad \tilde{s}_{ij}^n(z_k^n) &= \tilde{s}_{ij}^n(z_p^{n-1}) + \varepsilon^n \tilde{s}_{ij}^{(1)}(z_q^{(n)}) + O(\varepsilon^{n+1}), \\
 \tilde{\mu}_{ij}^n(z_k^n) &= \tilde{\mu}_{ij}^n(z_p^{n-1}) + \varepsilon^n \tilde{\mu}_{ij}^{(1)}(z_q^{(n)}) + O(\varepsilon^{n+1}); \\
 (3.9) \quad \tilde{\sigma}_k^n(h_j^n) &= \tilde{\sigma}_k^n(h_j^{n-1}) + \varepsilon^n I_{kp} \ddot{\varphi}_p^{(n)} + O(\varepsilon^{n+1}).
 \end{aligned}$$

Now taking the divergence of (3.8) and using (2.15), (3.6) and (3.9) we obtain that (3.7) is a solution of n -th-order theory of micropolar elasticity.

We may summarize the procedure as follows.

i) Assume that a solution $(u_j^{n-1}, \varphi_j^{n-1})$ of the equations $n-1$ -order micropolar elasticity has been found. Substituting $(u_j^{n-1}, \varphi_j^{n-1})$ into the equations of n -th-order micropolar elasticity, find the loads $(-\varepsilon^n \tilde{F}_i^{(n)}, -\varepsilon^n \tilde{G}_j^{(n)}, -\varepsilon^n \tilde{f}_k^{(n)}, -\varepsilon^n \tilde{g}_p^{(n)})$ required to maintain the displacement and microrotation in equilibrium.

ii) Find the solution $(\varepsilon^n u_i^{(n)}, \varepsilon^n \varphi_i^{(n)})$ of the infinitesimal theory corresponding to the reversed loads $(\varepsilon^n \tilde{F}_i^{(n)}, \varepsilon^n \tilde{G}_j^{(n)}, \varepsilon^n \tilde{f}_k^{(n)}, \varepsilon^n \tilde{g}_p^{(n)})$.

iii) Then a solution of the equations of n -th-order micropolar elasticity is given by $(u_i^{n-1} + \varepsilon^n u_i^{(n)}, \varphi_j^{n-1} + \varepsilon^n \varphi_j^{(n)})$.

In this method the step from any stage to the next is a small one and hence nearly infinitesimal. The discrepancy of a solution the $n-1$ stage from being one at the n -th stage is equivalent to the effect of a set of loads which are cancelled by supplying the solution that corresponds, according to the infinitesimal theory, to their negatives.

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METODE SISTEMATICE DE APROXIMARE ÎN TEORIA NELINIARĂ A MEDIILOR COSSERAT (III)

(Rezumat)

Se dă o metodă de perturbare de tip Signorini pentru problema la limită în tensiuni și micromomente în elastodinamica finită a mediilor micropolare. Se generalizează rezultatele lui Rivlin și Topaloglu la teoria neliniară a mediilor Cosserat.

It is found that a solution of 2.5 g of the substance in 10 ml of water gives a clear solution. The solution is colorless and has a pH of 7.0. The substance is soluble in water, alcohol, and ether. It is insoluble in benzene, chloroform, and carbon tetrachloride.

(b) The substance is a white, crystalline solid. It melts at 120°C and boils at 250°C. It is stable in air and does not decompose.

(c) The substance is a white, crystalline solid. It melts at 120°C and boils at 250°C. It is stable in air and does not decompose.

(d) The substance is a white, crystalline solid. It melts at 120°C and boils at 250°C. It is stable in air and does not decompose.

(e) The substance is a white, crystalline solid. It melts at 120°C and boils at 250°C. It is stable in air and does not decompose.

(f) The substance is a white, crystalline solid. It melts at 120°C and boils at 250°C. It is stable in air and does not decompose.

(g) The substance is a white, crystalline solid. It melts at 120°C and boils at 250°C. It is stable in air and does not decompose.

(h) The substance is a white, crystalline solid. It melts at 120°C and boils at 250°C. It is stable in air and does not decompose.

(i) The substance is a white, crystalline solid. It melts at 120°C and boils at 250°C. It is stable in air and does not decompose.

(j) The substance is a white, crystalline solid. It melts at 120°C and boils at 250°C. It is stable in air and does not decompose.

(k) The substance is a white, crystalline solid. It melts at 120°C and boils at 250°C. It is stable in air and does not decompose.

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(o) The substance is a white, crystalline solid. It melts at 120°C and boils at 250°C. It is stable in air and does not decompose.

(p) The substance is a white, crystalline solid. It melts at 120°C and boils at 250°C. It is stable in air and does not decompose.

(q) The substance is a white, crystalline solid. It melts at 120°C and boils at 250°C. It is stable in air and does not decompose.

(r) The substance is a white, crystalline solid. It melts at 120°C and boils at 250°C. It is stable in air and does not decompose.

(s) The substance is a white, crystalline solid. It melts at 120°C and boils at 250°C. It is stable in air and does not decompose.

(t) The substance is a white, crystalline solid. It melts at 120°C and boils at 250°C. It is stable in air and does not decompose.

(u) The substance is a white, crystalline solid. It melts at 120°C and boils at 250°C. It is stable in air and does not decompose.

(v) The substance is a white, crystalline solid. It melts at 120°C and boils at 250°C. It is stable in air and does not decompose.

(w) The substance is a white, crystalline solid. It melts at 120°C and boils at 250°C. It is stable in air and does not decompose.

(x) The substance is a white, crystalline solid. It melts at 120°C and boils at 250°C. It is stable in air and does not decompose.

(y) The substance is a white, crystalline solid. It melts at 120°C and boils at 250°C. It is stable in air and does not decompose.

THE SEMICONDUCTOR JUNCTION WITH POSITION DEPENDENT BAND STRUCTURE

BY

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1. Introduction. The semiconductors with position-dependent band structure have a position-dependent band gap E_g , bottom of the conduction band E_c , top of the valence band E_v , density of states g and electron affinity χ . A graded composition of two or more components, a dependence on position of the impurity doping, a nonuniform strain or a nonuniform temperature determine a nonuniform band structure. Many authors approach the problem of the semiconductors with position-dependent band structure [1] ... [11].

In a heterojunction, at the contact of n -semiconductor with p -semiconductors there is a region with a graded composition. Therefore, this zone has a position-dependent band structure.

The electron-phonon interaction is of a particular importance when one of the two systems is not at equilibrium. If the phonon system is perturbed by application of a temperature gradient, the phonons acquire a drift velocity. Because of the phonon-electron interaction the electrons acquire also a drift velocity parallel to those of the electron. This is the effect of carriers drag by phonons. The application of an electric field leads to a deviation of the electron distribution function from its equilibrium value and the electrons acquire a drift velocity. The phonon distribution is also modified through the electron-phonon interaction and the phonons acquire a drift velocity parallel to those of the electrons. This effect is called the phonons drag by electrons.

The drag phenomenon is important at low temperature, when dominates the interaction of electron with acoustic phonons (the deviation of the phonon distribution function become considerable when the phonon lifetime is long—this is the case of the acoustic phonons at low temperatures).

In this paper we report a theoretical study of the $n-p$ junction. We consider a semiconductor junction with a temperature gradient at low temperature and we take into account the electron drag by phonons.

2. The Density of Current, When a temperature gradient is applied to a semiconductor, the phonon and the electron distributions are

$$(1) \quad N(\mathbf{q}, \mathbf{r}) = N_0(\mathbf{q}, \mathbf{r}) + N_{an}(\mathbf{q}, \mathbf{r}) = N_0(\mathbf{q}, \mathbf{r}) + q_T N_1(\mathbf{q}, \mathbf{r}),$$

$$(2) \quad f(\mathbf{k}, \mathbf{r}) = f_0(\mathbf{k}, \mathbf{r}) + f_{an}(\mathbf{k}, \mathbf{r}) = f_0(\mathbf{k}, \mathbf{r}) + k_T f_1(\mathbf{k}, \mathbf{r}),$$

where $N_0(\mathbf{q})$ and $f_0(\mathbf{k}, \mathbf{r})$ are the distribution at equilibrium, $N_{an}(\mathbf{q})$ and $f_{an}(\mathbf{k}, \mathbf{r})$ — the anisotropic parts, q_T and k_T — the components of the electron, respectively phonon wave vector in the temperature gradient direction. We take into account, thus, the carrier drag by phonons in the relation (2).

At low temperature, the phonon-phonon interaction and the boundary scattering of phonons may be neglected. We consider only the interaction of electrons with acoustic phonons and with ionised impurities. The Boltzmann equation for the phonons and electrons are [12]

$$(3) \quad \mathbf{r} \nabla_{\mathbf{r}} N = \frac{\partial N}{\partial t} \Big|_{\text{coll}},$$

$$(4) \quad \frac{\partial f}{\partial t} + \mathbf{v}_{\mathbf{k}} \nabla_{\mathbf{r}} f - \frac{1}{\hbar} (\nabla E_c + \nabla W) \nabla_{\mathbf{k}} f = \hat{C}_a f + \hat{C}_I f,$$

where W is the electron kinetic energy, $\hat{C}_a$ and $\hat{C}_I$ — the collision operators for the electron scattering processes on the acoustic phonons and respectively on the impurities.

Following the method of N. Perrin and H. Budd [13], [14] we retain in the calculation of $\hat{C}_a f$ only the linear terms in k_T and q_T

$$(5) \quad \begin{aligned} \hat{C}_a f = & \frac{\pi C_1^2}{V \rho c_s} \sum_{\mathbf{q}} q \left\{ [f(\mathbf{k} + \mathbf{q}, \mathbf{r})(N(\mathbf{q}, \mathbf{r}) + 1) - f(\mathbf{k}, \mathbf{r})N(\mathbf{q}, \mathbf{r})] \times \right. \\ & \times \delta[E(\mathbf{k} + \mathbf{q}, \mathbf{r}) - E(\mathbf{k}, \mathbf{r}) - \hbar \omega_{\mathbf{q}}] + [f(\mathbf{k} - \mathbf{q}, \mathbf{r})N(\mathbf{q}, \mathbf{r}) - \\ & \left. - f(\mathbf{k}, \mathbf{r})(N(\mathbf{q}, \mathbf{r}) + 1)] \delta[E(\mathbf{k}, \mathbf{r}) - E(\mathbf{k} - \mathbf{q}, \mathbf{r}) - \hbar \omega_{\mathbf{q}}] \right\}, \end{aligned}$$

$$(6) \quad \hat{C}_a(k_T f_1) = -k_T \left[\frac{f_1(\mathbf{k}, \mathbf{r})}{\tau_a} + \frac{f_0(\mathbf{k}, \mathbf{r})}{x} \right];$$

C_1 is the total deformation potential constant, c_s — the longitudinal sound velocity, τ_a — the electron-acoustic phonon scattering relaxation time when the anisotropic part of the distribution function is not take into account. Here, x is given by the relation

$$(7) \quad \frac{1}{x} = \frac{C_1^2 m^*}{4\pi \rho \hbar k_B T_e} \int_0^{2\hbar} N_1(\mathbf{q}, \mathbf{r}) q^2 dq.$$

The interaction of electrons with ionised impurities is given by

$$(8) \quad \hat{C}_I(k_E f_1) = - \frac{k_T f_1(\mathbf{k}, \mathbf{r})}{\tau_I}.$$

For a stationary state, introducing (6) and (8) in (4) and taking into account the expression for f_0 it results

$$(9) \quad f_1 = \frac{\tau}{k_T} \mathbf{v}_k \frac{\partial f_0}{\partial E} (1+L) \nabla F_{fn} + \frac{\tau}{k_T} \mathbf{v}_k \frac{E-E_{fn}}{T} \frac{\partial f_0}{\partial E} \nabla T,$$

where

$$(10) \quad \frac{1}{\tau} = \frac{1}{\tau_a} + \frac{1}{\tau_I},$$

$$(11) \quad L = - \frac{k_T f_0}{x \mathbf{v}_k \nabla E_{fn} \frac{\partial f_0}{\partial E}};$$

E_{fn} is the quasi-Fermi level.

The electron density of current is

$$(12) \quad \mathbf{j}_n = - \frac{e}{4\pi^3} \int \mathbf{v}_k f_1 d^3k = n \mu_n \Delta E_{fn} - \frac{L_{ns}}{T} \nabla T,$$

with the notations

$$(13) \quad \mu_n = - \frac{e}{4\pi^3 n} \int \frac{\tau \mathbf{v}_k \mathbf{v}_k}{k_T} \frac{\partial f_0}{\partial E} (1+L) d^3k,$$

$$(14) \quad L_{ns} = \frac{e}{4\pi^3} \int \frac{\tau \mathbf{v}_k \mathbf{v}_k}{k_T} (E-E_{fn}) \frac{\partial f_0}{\partial E} d^3k,$$

where μ_n is the electron mobility in semiconductors with position dependent band structure if we take into account the carrier drag by phonons.

The quantity E_{fn} may be written under the form [5]

$$(15) \quad \nabla E_{fn} = \nabla E_c + \left(\frac{\partial E_{fn}}{\partial n} \right)_{T,\epsilon} \nabla n + \left(\frac{\partial E_{fn}}{\partial T} \right)_{n,\epsilon} \nabla T - \left(\frac{\partial E_{fn}}{\partial n} \right)_{T,\epsilon} \frac{1}{4\pi^3} \int d^3k \frac{\partial f_0}{\partial E} \nabla W;$$

E_c and χ are bounded by the relation

$$(16) \quad E_c = E_0 - e\varphi - \chi,$$

where E_0 is the zero level of vacuum and φ — the electric potential. Considering only a x -dependence, from the relations (12), (15) and (16) we obtain

$$(17) \quad j_n = n \mu_n \left[-e \frac{d\varphi}{dx} - \frac{d\chi}{dx} + \left(\frac{\partial E_{fn}}{\partial n} \right)_{T,\epsilon} \frac{dn}{dx} + \left(\frac{\partial E_{fn}}{\partial T} \right)_{n,\epsilon} \frac{dT}{dx} - \left(\frac{\partial E_{fn}}{\partial n} \right)_{T,\epsilon} \frac{1}{4\pi^3} \int d^3k \frac{\partial f_0}{\partial E} \frac{dW}{dx} \right] - \frac{L_{ns}}{T} \frac{dT}{dx}.$$

For the hole density, analogically it results

$$(18) \quad j_p = p \mu_p \left[-e \frac{d\varphi}{dx} - \frac{dE_g}{dx} - \frac{d\chi}{dx} + \left(\frac{\partial E_{fp}}{\partial p} \right)_{T,\epsilon} \frac{dp}{dx} + \left(\frac{\partial E_{fp}}{\partial T} \right)_{p,\epsilon} \frac{dT}{dx} - \left(\frac{\partial E_{fp}}{\partial p} \right)_{T,\epsilon} \frac{1}{4\pi^3} \int d^3k \frac{\partial f_h^0}{\partial E_h} \frac{\partial W_h}{\partial x} \right] - \frac{L_{ps}}{T} \frac{dT}{dx}.$$

The total density of current is

$$(19) \quad j = j_n + j_p = n\mu_n A_n + p\mu_p A_p - \frac{1}{T} (L_{ns} + L_{ps}) \frac{dT}{dx}.$$

We noted with A_n and A_p the square bracket in (17) and (18).

3. Conclusions. 1. The form of the electron current density in the semiconductor with position-dependent band structure and taking into account the drag phenomenon is the same with the expression of the electron current density when electron drag by phonons is not taken into account [5]. But the electron mobility is different. The result of A. Marshak and K.M. van Vliet is

$$(20) \quad \mu_n = - \frac{e}{4\pi^3 n} \int \tau \mathbf{v}_k \mathbf{v}_k \frac{\partial f_0}{\partial E} d^3 k.$$

Comparing (20) with (13) we notice in our case, the appearance of the factor $(1+L)/k_T$. Thus, the drag phenomenon modifies the electron and hole mobility.

2. The relation (19) is the general expression for the density of current in a $n-p$ junction with a temperature gradient and nonuniform band structure. For a concrete calculation of j we must know the x -dependence of χ , E_g , W , W_n and T .

3. From (17) by integration between the limits of the transition region (x_n , x_p) we obtain the connection relation between the depletion layer diffusion potential V_i and the carrier densities $n(x_n)$ and $n(x_p)$.

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JONCTIUNEA SEMICONDUCTOARE CU STRUCTURĂ DE BANDĂ DEPENDENTĂ DE POZIȚIE

(Rezumat)

Se studiază densitatea de curent într-o joncțiune semiconductoră cu structură de bandă dependentă de poziție. Se consideră că joncțiunea este supusă unui gradient de temperatură și se ia în considerare antrenarea purtătorilor de sarcină de către fononi.

AN ORDER RELATION

BY

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Let M be a metric space with the distance d and $\mathbb{R}$ be the set of reals. The purpose of the paper is to endow $M \times \mathbb{R}$ with an order relation and then to apply it to some special sets.

In order to do this we consider an arbitrary function $F: M \rightarrow \mathbb{R}$ and an injective function $f: \mathbb{R} \rightarrow \mathbb{R}$. We fix two numbers $k_1 \geq 0$ and $k_2 > 0$ and define on $M \times \mathbb{R}$ the following relation

$$(1) \quad (a, x) \leq (b, y) \Leftrightarrow d(b, a) \leq k_1(F(b) - F(a)) + k_2(f(y) - f(x)).$$

Proposition 1. *The set $M \times \mathbb{R}$ is partially ordered by the relation (1).*

Proof. I. According to $d(a, a) = 0 = k_1(F(a) - F(a)) + k_2(f(x) - f(x))$, we have $(a, x) \leq (a, x)$ for every $(a, x) \in M \times \mathbb{R}$.

II. If $(a, x) \leq (b, y)$ and $(b, y) \leq (a, x)$ then $(a, x) = (b, y)$ that is $a = b$ and $x = y$. In fact, $(a, x) \leq (b, y)$ and $(b, y) \leq (a, x) \Leftrightarrow d(a, b) \leq k_1(F(b) - F(a)) + k_2(f(y) - f(x))$ and $d(b, a) \leq k_1(F(a) - F(b)) + k_2(f(x) - f(y))$. But $d(a, b) = d(b, a) \geq 0$, therefore $k_1(F(b) - F(a)) + k_2(f(y) - f(x)) \geq 0$ and $k_1(F(b) - F(a)) + k_2(f(y) - f(x)) \leq 0 \Rightarrow k_1(F(b) - F(a)) + k_2(f(y) - f(x)) = 0 \Rightarrow 0 \leq d(a, b) \leq 0 \Rightarrow d(a, b) = 0 \Rightarrow a = b \Rightarrow k_2(f(y) - f(x)) = 0 \Rightarrow f(y) = f(x) \Rightarrow x = y$.

III. If $(a, x) \leq (b, y)$ and $(b, y) \leq (c, z)$ then $(a, x) \leq (c, z)$. In fact, we have

$$d(a, b) \leq k_1(F(b) - F(a)) + k_2(f(y) - f(x))$$

and

$$d(b, c) \leq k_1(F(c) - F(b)) + k_2(f(z) - f(y)).$$

But $d(a, c) \leq d(a, b) + d(b, c) \leq k_1(F(c) - F(a)) + k_2(f(z) - f(x)) \Leftrightarrow (a, x) \leq (c, z)$.

Remark. For $F=0$ or $k_1=0$, and $k_2=1$ and f =the identity, we obtain the relation of order in [2] as a particular case.

Proposition 2. *Let F be an endomorphism and f be an injective endomorphism of the additive group $\mathbb{R}$. Then the $\mathbb{R} \times \mathbb{R}$ is an ordered group with respect to the operation $(a_1, b_1) + (a_2, b_2) = (a_1 + a_2, b_1 + b_2)$ and order relation $(a_1, b_1) \leq (a_2, b_2) \Leftrightarrow |a_1 - a_2| \leq k_1(F(a_2) - F(a_1)) + k_2(f(b_2) - f(b_1))$, where $k_1 \geq 0$ and $k_2 > 0$ are fixed in $\mathbb{R}$.*

Proof. By virtue of Proposition 1, we only prove that if $(a_1, b_1) \leq (a_2, b_2)$ then $(a_1, b_1) + (a, b) = (a_1 + a, b_1 + b) \leq (a_2, b_2) + (a, b) = (a_2 + a, b_2 + b)$ for every $(a, b) \in \mathbb{R} \times \mathbb{R}$. But $(a_1 + a, b_1 + b) \leq (a_2 + a, b_2 + b) \Leftrightarrow |a_2 + a - a_1 - a| = |a_2 - a_1| \leq k_1(F(a_2 + a) - F(a_1 + a)) + k_2(f(b_2 + b) - f(b_1 + b)) = k_1(F(a_2) - F(a_1)) + k_2(f(b_2) - f(b_1))$ which is true.

Proposition 3. If $0 < k \leq 1$, then $\mathbf{R} \times \mathbf{R}$ becomes a partially ordered ring by the operations $(a_1, b_1) + (a_2, b_2) = (a_1 + a_2, b_1 + b_2)$, $(a_1, b_1)(a_2, b_2) = (a_1 a_2, b_1 b_2)$ with the order given by $(a_1, b_1) \leq (a_2, b_2) \Leftrightarrow |a_1 - a_2| \leq k(b_2 - b_1)$.

Proof. By virtue of Propositions 1 and 2 we only prove that if $(a_1, b_1) \leq (a_2, b_2)$ and if $(a, b) \geq (0, 0)$, then $(aa_1, bb_1) \leq (aa_2, bb_2)$.

By hypotheses we have $|a_2 - a_1| \leq k(b_2 - b_1)$ and $|a| \leq kb$. Hence, $|a_2 - a_1||a| = |aa_2 - aa_1| \leq k^2(bb_2 - bb_1) \leq k(bb_2 - bb_1)$, that is, $(aa_1, bb_1) \leq (aa_2, bb_2)$.

In the Propositions 4 and 5, we suppose that $\mathbf{R} \times \mathbf{R}$ is a partially ordered group as in Proposition 2 (that is the conditions of Proposition 2 are fulfilled).

Proposition 4. If G is an additive group and g is a homomorphism from G to $\mathbf{R} =$ the multiplicative group of the non-zero reals, then occur:

1. $H = G \times (\mathbf{R} \times \mathbf{R})$ becomes a group by the operation $(a_1, x_1, y_1) \circ (a_2, x_2, y_2) = (a_1 + a_2, g(a_1)x_2 + x_1, g(a_1)y_2 + y_1)$.

2. If G is a left (right) ordered group then H is a left (right) ordered group by the cone $P = \{(a, x, y) ; a > 0 \text{ or } a = 0 \text{ and } (x, y) \geq (0, 0)\}$.

3. H is a left (right) ordered group by the cone $P = \{(a, x, y) ; g(a) > 1 \text{ or } a = 0 \text{ and } (x, y) \geq (0, 0)\}$.

4. If G' is an additive left (right) ordered group and h is a homomorphism from G to G' , then, H is left (right) ordered group by the cone $P = \{(a, x, y) ; h(a) > 0 \text{ or } a = 0 \text{ and } (x, y) \geq (0, 0)\}$.

Proof of 1. I. We verify the associativity.

$$[(a_1, x_1, y_1) \circ (a_2, x_2, y_2)] \circ (a_3, x_3, y_3) = (a_1 + a_2, g(a_1)x_2 + x_1, g(a_1)y_2 + y_1) \circ (a_3, x_3, y_3) = (a_1 + a_2 + a_3, g(a_1 + a_2)x_3 + g(a_1)x_2 + x_1, g(a_1 + a_2)y_3 + g(a_1)y_2 + y_1).$$

$$(a_1, x_1, y_1) \circ [(a_2, x_2, y_2) \circ (a_3, x_3, y_3)] = (a_1, x_1, y_1) \circ (a_2 + a_3, g(a_2)x_3 + x_2, g(a_2)y_3 + y_2).$$

$$g(a_2)y_3 + y_2 = (a_1 + a_2 + a_3, g(a_1)(g(a_2)x_3 + x_2) + x_1, g(a_1)(g(a_2)y_3 + y_2) + y_1).$$

According to the fact that g is a homomorphism, the associativity is fulfilled.

II. $\exists (0, 0, 0) \in H$ such that $\forall (a, x, y) \in H$ we have $(a, x, y) \circ (0, 0, 0) = (a, g(a)0 + x, g(a)0 + y) = (a, x, y)$.

III. If $(a, x, y) \in H$, $\exists (-a, g(-a)(-x), g(-a)(-y)) \in H$ such that $(a, x, y) \circ (-a, g(-a)(-x), g(-a)(-y)) = (0, g(a)g(-a)(-x) + x, g(a)g(-a)(-y) + y) = (0, 0, 0)$.

IV. Generally, the group H is not commutative. In fact, $(a_1, x_1, y_1) \circ (a_2, x_2, y_2) = (a_1 + a_2, g(a_1)x_2 + x_1, g(a_1)y_2 + y_1)$ is generally different of $(a_2, x_2, y_2) \circ (a_1, x_1, y_1) = (a_2 + a_1, g(a_2)x_1 + x_2, g(a_2)y_1 + y_2)$.

Proof of 2. $P = \{(a, x, y) ; a > 0 \text{ or } a = 0 \text{ and } (x, y) \geq (0, 0)\}$.

I. $P \circ P \subseteq P$. Let $(a_i, x_i, y_i) \in P$, $i = \overline{1, 2}$ and $\gamma = (a_1, x_1, y_1) \circ (a_2, x_2, y_2) = (a_1 + a_2, g(a_1)x_2 + x_1, g(a_1)y_2 + y_1)$.

If $a_1 > 0$ and $a_2 > 0$ then $a_1 + a_2 > 0$ and $\gamma \in P$.

If $a_1 = 0$ and $(x_1, y_1) \geq (0, 0)$ and $a_2 > 0$ then $a_1 + a_2 > 0$ and $\gamma \in P$.

If $a_2=0$ and $(x_2, y_2) \geq (0, 0)$ and $a_1 > 0$ then $a_1 + a_2 > 0$ and $\gamma \in P$.

If $a_1 = a_2 = 0$ and $(x_1, y_1) \geq (0, 0)$ and $(x_2, y_2) \geq (0, 0)$ then $a_1 + a_2 = 0$ and $(x_2 + x_1, y_2 + y_1) \geq (0, 0)$ and then $\gamma \in P$.

II. $P \cap (-P) = (0, 0, 0)$. Let $(a, x, y) \in P \cap (-P)$, hence, $a > 0$ or $(a = 0$ and $(x, y) \geq (0, 0))$ and $(a, x, y) = (-a', g(-a')(-x'), g(-a')(-y'))$ with $a' > 0$ or $(a' = 0$ and $(x', y') \geq (0, 0))$.

We cannot have $a > 0$, $a = -a'$ and $a' > 0$.

We cannot have $a > 0$, $a = -a'$, $a' = 0$ and $(x', y') \geq (0, 0)$.

We cannot have $a = 0$, $(x, y) \geq (0, 0)$, $a = -a'$ and $a' > 0$.

It remains the case $a = 0$, $(x, y) \geq (0, 0)$, $a' = 0$, $(x', y') \geq (0, 0)$, $x = -x'$, $y = -y'$. We have $(0, 0) \leq (x, y)$ and $(0, 0) \leq (-x, -y) \Rightarrow |x| \leq k_1 F(x) + k_2 f(y)$ and $|-x| = |x| \leq k_1 F(-x) + k_2 f(-y) \Rightarrow k_1 F(x) + k_2 f(y) \geq 0$ and $k_1 F(x) + k_2 f(y) \leq 0 \Rightarrow k_1 F(x) + k_2 f(y) = 0 \Rightarrow 0 \leq |x| \leq 0 \Rightarrow x = 0 \Rightarrow k_2 f(y) = 0 \Rightarrow f(y) = 0 = f(0) \Rightarrow y = 0 \Rightarrow (a, x, y) = (0, 0, 0)$.

The proofs for 3 and 4 follow in the same way.

Proposition 5. If G is an additive group and g is a homomorphism from G to $\mathbb{R}_+^*$ = the multiplicative group of the positive reals and if k_1 in Proposition 2 is zero or F in Proposition 2 has the property that for every $b \in G$, $F(g(b)x) = g(b)F(x)$, and f in Proposition 2 has the property that for every $b \in G$, $f(g(b)y) = g(b)f(y)$, then occur:

1. $H = G \times (\mathbb{R} \times \mathbb{R})$ becomes a group by the operation $(a_1, x_1, y_1) \circ (a_2, x_2, y_2) = (a_1 + a_2, g(a_1)x_2 + x_1, g(a_1)y_2 + y_1)$.

2. If G is a partially ordered group, then H is a partially ordered group by the cone $P = \{(a, x, y) ; a > 0 \text{ or } a = 0 \text{ and } (x, y) \geq (0, 0)\}$.

3. H is a partially ordered group by the cone

$$P = \{(a, x, y) ; g(a) > 1 \text{ or } a = 0 \text{ and } (x, y) \geq (0, 0)\}.$$

4. If G' is an additive partially ordered group and h is a homomorphism from G to G' , then H is a partially ordered group by the cone $P = \{(a, x, y) ; h(a) > 0 \text{ or } a = 0 \text{ and } (x, y) \geq (0, 0)\}$.

Proof. The assertion 1 here is the same with the assertion 1 from Proposition 4. 2. By virtue of the assertion 2 in Proposition 4, we only prove that if $(b, u, v) \in H$ and $(a, x, y) \in P$, then $\delta = (b, u, v) \circ (a, x, y) \circ (-b, g(-b)(-u), g(-b)(-v)) \in P$, that is $h \circ P \circ (-h) \subseteq P$ for every $h \in H$.

$$\begin{aligned} \delta &= (b + a, g(b)x + u, g(b)y + v) \circ (-b, g(-b)(-u), g(-b)(-v)) = \\ &= (b + a - b, g(b + a)g(-b)(-u) + g(b)x + u, g(b + a)g(-b)(-v) + g(b)y + v). \end{aligned}$$

If $a > 0$, $b + a - b > 0$ and then $\delta \in P$.

If $a = 0$ and $(x, y) \geq (0, 0) \Leftrightarrow a = 0$ and $|x| \leq k_1 F(x) + k_2 f(y)$, then, $b + a - b = 0$ and we must see if $(g(b)x, g(b)y) \geq (0, 0)$ that is if $|g(b)x| = g(b)|x| \leq k_1 F(g(b)x) + k_2 f(g(b)y) = k_1 g(b)F(x) + k_2 g(b)f(y)$ which is true.

The proofs of 3 and 4 follow in the same way.

Remark. The Propositions 4 and 5 can be considered as parts of a more general outline presented in Theorem 1 [1].

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O RELAȚIE DE ORDINE

(Rezumat)

Dacă M este un spațiu metric, atunci pe $M \times \mathbb{R}$ se introduce o relație de ordine, cu ajutorul căreia pe $\mathbb{R} \times \mathbb{R}$ și pe alte mulțimi se obțin unele structuri algebrice ordonate.

ABOUT A 2^{n+1} -ORDER ALGEBRA

BY

DANIEL CONDURACHE

1. Let be $\mathcal{A}_k, k=\overline{0, n}, n+1$ second order algebras over the real number field isomorphic with the complex field. if $\{\hat{1}_k, \hat{i}_k\}, k=\overline{0, n}$, are elements of a basis in $\mathcal{A}_k$, the multiplication table is written as

$$(1) \quad \begin{aligned} \hat{1}_k \hat{1}_k &= \hat{1}_k \quad \hat{1}_k \hat{i}_k = \hat{i}_k, \\ i_k^2 &= -\hat{1}_k^2 = -\hat{1}_k. \end{aligned}$$

We consider algebra $\mathcal{A} = \mathcal{A}_0 \odot \mathcal{A}_1 \odot \dots \odot \mathcal{A}_n$ as a direct product of algebras $\mathcal{A}_k, k=\overline{0, n}$. Algebra $\mathcal{A}$ is an associative and commutative one and it has a unit element, ord $\mathcal{A} = 2^{n+1}$. With the writing convention

$$(2) \quad \hat{i}_k^{p_k} = \begin{cases} \hat{1}_k & \text{if } p_k=0, \\ \hat{i}_k & \text{if } p_k=1, \end{cases}$$

elements

$$(3) \quad \varepsilon_p = \hat{i}_0^{p_0} \hat{i}_1^{p_1} \dots \hat{i}_n^{p_n}, \quad p = \overline{0, 2^{n+1}-1},$$

where $p = (p_0 p_1 \dots p_n)_2$ are the writing of p in the second basis with $n+1$ bits, constitute the elements of a basis in $\mathcal{A}$. From (1), (2) and (3), the following multiplication table in $\mathcal{A}$ results:

$$(4) \quad \hat{\varepsilon}_p \hat{\varepsilon}_q = (-1)^{\langle p, q \rangle} \varepsilon_{p \oplus q},$$

where $\langle p, q \rangle = p_0 q_0 + p_1 q_1 + \dots + p_n q_n$ and $p \oplus q$ is the modulo 2 addition bit by bit [1] of numbers p and q . $\hat{\varepsilon}_0$ is the unit of algebra $\mathcal{A}$.

Function $g: \{0, 1, \dots, 2^{n+1}-1\} \rightarrow \{0, 1, \dots, 2^{n+1}-1\}$ defined by means of

$$(5) \quad g(k) = (\bar{k}_0, \bar{k}_1, \dots, \bar{k}_n)_2,$$

where

$$\bar{k}_p = \begin{cases} k_0 & \text{if } p=n, \\ k_{n-p} \oplus k_{n-p-1} & \text{if } p \in \{0, 1, \dots, n-1\} \end{cases}$$

$$k = (k_0 k_1 \dots k_n)_2$$

sometimes denoted as „Gray coding” [8] is a bijective one. Elements

$$(6) \quad \hat{v}_k = \hat{\varepsilon}_{g(k)} = \hat{1}_0^{\bar{p}_0} \hat{1}_1^{\bar{p}_1} \dots \hat{1}_n^{\bar{p}_n}, \quad k \in \{0, 1, \dots, 2^{n+1} - 1\}$$

also form a basis in $\mathcal{A}$. As in Gray code with $n+1$ bits, numbers $0, 1, \dots, 2^n - 1$ are written with an odd number of bits of one, while numbers $2^n, 2^n + 1, \dots, 2^{n+1} - 1$ are written with even ones, it results

$$(7) \quad \hat{v}_k^2 = \begin{cases} \hat{v}_0 & \text{if } k \in \{0, 1, \dots, 2^n - 1\}, \\ -\hat{v}_0 & \text{if } k \in \{2^n, 2^n + 1, \dots, 2^{n+1} - 1\}; \\ \hat{v}_0 = \hat{\varepsilon}_0 = \hat{1}_0 \hat{1}_1 \dots \hat{1}_n. \end{cases}$$

To simplify the writing, we shall denote element of basis $\hat{1}_0 \hat{1}_1 \dots \hat{1}_{k-1} \hat{1}_k \hat{1}_{k+1}, \dots, \hat{1}_n$ as $\hat{1}_k$ and shall identify the real number field $\mathbf{R}$, as being $\mathcal{A}$ sub-algebra isomorphic with $\mathbf{R}$.

2. A previous paper [6] has demonstrated

Proposition 1. *Elements*

$$(8) \quad \hat{e}_p = \frac{1}{2^n} \prod_{v=1}^n [\hat{v}_0 - (-1)^{\hat{p}_v \hat{1}_0} \hat{v}_v], \quad p \in \{0, 1, \dots, 2^n - 1\},$$

where $p = (p_1 p_2 \dots p_n)_2$ constitute the writing of p with n bits in the second basis are 2^n orthogonal idempotentes in $\mathcal{A}$

$$(9) \quad \hat{e}_p \hat{e}_q = \begin{cases} \hat{e}_p & \text{if } p = q, \\ 0 & \text{if } p \neq q. \end{cases}$$

Proposition 2. For $(\forall) w_k \in \mathbf{R}$, $k = \overline{0, n}$, relation

$$(10) \quad \sum_{k=0}^n w_k \hat{1}_k = \sum_{p=0}^{2^n-1} \Omega_p \hat{1}_0 \hat{1}_p$$

takes place, where

$$\Omega_p = w_0 + \sum_{v=1}^n (-1)^{\hat{p}_v w_v}, \quad p = (p_1 p_2 \dots p_n)_2.$$

Let be

$$(11) \quad \sigma(k) = (-1)^{\left[\frac{\bar{k}_0 + \bar{k}_1 + \dots + \bar{k}_n}{2} \right]}; \quad k = \overline{0, 2^{n+1} - 1}, \quad g(k) = (\bar{k}_0 \bar{k}_1 \dots \bar{k}_n)_2,$$

where $[a]$ denotes the integer part of number a , and $\mathbf{H}_w$ the 2^n order Walsh matrix [1]. Matrix $\mathbf{H}_w$ is obtained by permuting matrix Hadamard [1] lines, $\mathbf{H} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{\otimes n}$; $\otimes$ denotes raising to a power in the Kronecker product of matrices way. [1], [2] proves matrix $\mathbf{H}_w$ to be a non-singular one and

$$(12) \quad \mathbf{H}_w^{-1} = \frac{1}{2^n} \mathbf{H}^T.$$

The following 2^{n+1} order matrices, obviously non-singular are considered:

$$(13) \quad S_1 = \left[\begin{array}{c|c} \begin{array}{cc} \sigma(0) & 0 \\ 0 & \sigma(2^n-1) \end{array} & \begin{array}{c} 0 \\ 0 \end{array} \\ \hline \begin{array}{c} 0 \\ 0 \end{array} & \begin{array}{cc} 0 & \sigma(2^{n+1}-1) \\ \sigma(2^n) & 0 \end{array} \end{array} \right] = \left[\begin{array}{c|c} \begin{array}{cc} \sigma & 0 \\ 0 & \sigma' \end{array} & \begin{array}{c} 0 \\ 0 \end{array} \\ \hline \begin{array}{c} 0 \\ 0 \end{array} & \begin{array}{cc} 0 & \sigma' \end{array} \end{array} \right],$$

$$(14) \quad S_2 = \left[\begin{array}{c|c} H_w & 0 \\ \hline 0 & H_w \end{array} \right],$$

$$(15) \quad S = S_1 S_2 = \left[\begin{array}{c|c} \sigma \cdot H_w & 0 \\ \hline 0 & \sigma' \cdot H_w \end{array} \right].$$

From (12), (13), (14) and (15), matrix S results to be a non-singular one and

$$(16) \quad S^{-1} = \frac{1}{2^n} S^T.$$

Proposition 3. If $\mathbf{v}$ and $\mathbf{e}$ are the column matrices

$$(17) \quad \mathbf{v} = [\hat{v}_0, \hat{v}_1, \dots, \hat{v}_{2^{n+1}-1}]^T,$$

$$(18) \quad \mathbf{e} = [\hat{e}_0, \hat{e}_1, \dots, \hat{e}_{2^n-1}, \hat{i}_0 \hat{e}_0, \hat{i}_0 \hat{e}_1, \dots, \hat{i}_0 \hat{e}_{2^{n+1}-1}],$$

then

$$(19) \quad \mathbf{v} = \mathbf{S} \cdot \mathbf{e},$$

$$(20) \quad \mathbf{e} = \frac{1}{2^n} S^T \cdot \mathbf{v}.$$

Proof. Taking in (10) $w_k = \delta_{kp}$, δ_{kp} as being the Kronecker delta, we obtain

$$(21) \quad \hat{i}_k = \sum_{p=0}^{2^n-1} (-1)^{p_k} \hat{i}_0 \hat{e}_p, \quad k \in \{0, 1, \dots, n\}.$$

From (6), (21) and (9) it results

$$(22) \quad \hat{v}_k = \sum_{p=0}^{2^n-1} (-1)^{p_1 \bar{k}_1 + p_2 \bar{k}_2 + \dots + p_n \bar{k}_n} \hat{i}_0^{\bar{k}_0 + \bar{k}_1 + \dots + \bar{k}_n} \hat{e}_p.$$

Using observation which have followed to relations (7) and the fact that $\hat{i}_0^2 = -\hat{v}_0$, it results

$$(23) \quad \hat{i}_0^{\bar{k}_0 + \bar{k}_1 + \dots + \bar{k}_n} = \begin{cases} (-1)^{\left[\frac{\bar{k}_0 + \bar{k}_1 + \dots + \bar{k}_n}{2} \right]} \hat{v}_0 = \sigma(k) \hat{v}_0, & \text{if } k \in \{0, 1, \dots, 2^n-1\}, \\ (-1)^{\left[\frac{\bar{k}_0 + \bar{k}_1 + \dots + \bar{k}_n}{2} \right]} \hat{i}_0 = \sigma(k) \hat{i}_0, & \text{if } k \in \{2^n, 2^n+1, \dots, 2^{n+1}-1\}. \end{cases}$$

Element $a_{kp} = (-1)^{p_1\bar{k}_1 + p_2\bar{k}_2 + \dots + p_n\bar{k}_n}$ represents the one which is to be found in the foot of line k with column p in matrix $\mathbf{H}_w$ [1]. With this notice, from (23), (22) and (15) it results (19). From (19) and (16) it results (20).

3. Relations (19) and (20) demonstrate that elements of matrix $\mathbf{e}$ constitute a basis in algebra $\mathcal{A}$, also indicating the real way of a basis changing.

A certain element in $\mathcal{A}$

$$(24) \quad \hat{w} = \sum_{k=0}^{2^{n+1}-1} a_k \hat{v}_k; \quad a_k \in \mathbb{R}, \quad k = \overline{0, 2^{n+1}-1}$$

is written in basis $\mathbf{e}$ as

$$(25) \quad \hat{w} = \sum_{k=0}^{2^n-1} (b_k \hat{v}_0 + i_0 c_k) \hat{e}_k; \quad b_k, c_k \in \mathbb{R}, \quad k = \overline{0, 2^n-1}.$$

The real scalars b_k and c_k , $k = \overline{0, 2^n-1}$, are determined from (24), (19) and (15):

$$(26) \quad [b_0, b_1, \dots, b_{2^n-1}]^T = \sigma \cdot \mathbf{H}_w [a_0, a_1, \dots, a_{2^n-1}]^T,$$

$$(27) \quad [c_0, c_1, \dots, c_{2^n-1}]^T = \sigma' \cdot \mathbf{H}_w [a_{2^n}, a_{2^n+1}, \dots, a_{2^{n+1}-1}]^T.$$

As elements $\hat{e}_k$, $k = \overline{0, 2^n-1}$ represent 2^n orthogonal idempotents in algebra $\mathcal{A}$, relations (25) and well-known properties of algebras over the real number field [7] demonstrate

Proposition 4. Algebra $\mathcal{A}$ is a polynomial, semi-simple one being the direct sum of 2^n algebras isomorphic with the complex field.

4. Using relations (9), the properties of the exponential function defined over the complex field, the exponential form [3] for element (24) is obtained

$$(28) \quad \begin{aligned} \hat{w} &= \sum_{k=0}^{2^n-1} \exp[\ln(b_k \hat{v}_0 + i_0 c_k)] \hat{e}_k = \exp \sum_{k=0}^{2^n-1} \ln(b_k \hat{v}_0 + i_0 c_k) \hat{e}_k = \\ &= \exp \left[\sum_{k=0}^{2^n-1} \left(\ln \sqrt{b_k^2 + c_k^2} \hat{v}_0 + i_0 \arctg \frac{c_k}{b_k} \right) \hat{e}_k \right]; \quad b_k, c_k \neq 0, \quad k = \overline{0, 2^n-1}. \end{aligned}$$

If the exponential form of element $\hat{w}$ is

$$(29) \quad \hat{w} = \exp[\gamma_0 \hat{v}_0 + \dots + \gamma_{2^{n+1}-1} \hat{v}_{2^{n+1}-1}], \quad \gamma_k \in \mathbb{R}, \quad k = \overline{0, 2^{n+1}-1},$$

with $\hat{v}_0$ as the algebra unit, zero divisors in $\mathcal{A}$ may be characterized as [3]

$$(30) \quad N(\hat{w}) = \exp \gamma_0 = 0.$$

The first line of matrix $\mathbf{S}^{-1}$ (16) being $\left[\frac{1}{2^n}, \frac{1}{2^n}, \dots, \frac{1}{2^n}, 0, 0, \dots, 0 \right]$ from (28) it results

$$(31) \quad N(\hat{w}) = \frac{1}{2^n} \prod_{k=0}^{2^n-1} \sqrt{b_k^2 + c_k^2}.$$

From (30) and (31) we obtain

Proposition 5. *Element (24) is a zero divisor in $\mathcal{A}$, if there exists at least an $k \in \{0, 1, \dots, 2^n - 1\}$ so that the following equalities should take simultaneously place:*

$$(32) \quad b_k = c_k = 0.$$

We now suppose conditions

$$(33) \quad \begin{cases} b_p \neq b_q \neq 0 \\ c_p \neq c_q \neq 0 \quad (\forall) p, q \in \{0, 1, \dots, 2^n - 1\} \\ p \neq q \end{cases}$$

being simultaneously accomplished. Polynomial

$$(34) \quad p(\lambda) = \prod_{k=0}^{2^n-1} (\lambda^2 - 2b_k\lambda + b_k^2 + c_k^2)$$

in conditions (33) is a minimum annihilator one for element (24). This way takes place

Proposition 6. *The polynomial algebra $\mathcal{A}$ is generated by polynomial (34) in conditions (33); elements $b_k, c_k \in \mathbb{R}, k = \overline{0, 2^n - 1}$, being calculated by means of relations (26), (27).*

The elements of set $\{\hat{v}_0, \hat{w}, \dots, \hat{w}^{2^{n+1}-1}\}$ represent the power basis of algebra $\mathcal{A}$.

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ASUPRA UNEI ALGEBRE DE ORDIN 2^{n+1}

(Rezumat)

Se demonstrează caracterul de algebră polinomială al algebrei produs direct a $n+1$ algebre izomorfe cu corpul complex. Se precizează matricea schimbării de bază din baza produs direct în baza de idempotenți ortogonali, se caracterizează divizorii lui zero și se indică procedeul de construcție al polinomului generator.

BOUNDED AND DIRICHLET-FINITE HARMONIC FUNCTIONS ON STANDARD H -CONES

BY

LILIANA POPA

The main result of the paper is Theorem 6, which gives a characterization of the bounded substractible elements, having finite Dirichlet integral.

In the presence of some additional hypothesis we can define a Dirichlet integral for bounded elements of the H -cone, which is positive, [4]. The set of all bounded substractible elements with finite Dirichlet integral $\mathcal{H}_{BD}$ is a lattice with respect to the specific order, where „the module contraction operates“ (Theorem 2) and which is conditionally complete (Proposition 4).

In the sequel $S=S^{**}$ will be a standard H -cone of functions on a saturated set X , which is autodual relating to an H -morphism denoted by $\varphi: S \rightarrow S^*$. We suppose that the following hypothesis are fulfilled:

- i) $1 \in S_H$, where S_H is the set of all substractible elements of S [2];
- ii) S satisfies the axiom A [2];
- iii) S satisfies the axiom D [1];
- iv) X is locally connected with respect to the natural topology.

Let S_p be the set of all pure potentials of S [2]. We denote by

$$S_B = \{s \in S \mid \exists \alpha > 0, s \leq \alpha\}.$$

We recall from [4] that $[S_B] = S_B - S_B$ is an algebra relating to the usual product of functions. In the presence of the previous hypothesis we can define the following correspondence:

for any $s \in S_B$, $s = h + p$, $h \in S_H$, $p \in S_p$, there exists a non-negative measure on X , denoted by $\sigma(p)$, such that

$$p(x) = \int_X g_x(y) d\sigma(p)(y) \quad \text{for any } x \in X,$$

where $g_x(y)$ was defined in [4].

This map was called like in [3], measure of representation. We have shown [4] that for any $f, g \in [S_B]$, which are naturally continuous,

$$\delta_{[f,g]} = \frac{1}{2} \{f\sigma(g) + g\sigma(f) - \sigma(fg)\}$$

is a well defined measure, named the mutual gradient measure.

For any $f \in [S_B]$, naturally continuous,

$$\delta_f = \delta_{[f,f]} = f\sigma(f) - \frac{1}{2}\sigma(f^2) \geq 0.$$

If $h \in S_H \cap S_B$ and $p \in S_P \cap S_B$ and is naturally continuous, then $\delta_{[h,p]}(1) = 0$.

If $h \in S_H \cap S_B$, then $\delta_h = -\frac{1}{2}(h^2)$. Let

$$\mathcal{H}_{BD} = \{u = u_1 - u_2 \mid u_i \in S_H \cap S_B, \delta_{u_i}(1) < \infty, i = 1, 2\}.$$

From the next Lemma, we deduce that $\mathcal{H}_{BD}$ is a lattice with respect to the specific order.

Lemma 1. For any $u \in S_H \cap S_B$, we have $\delta_{[u \vee 0, u \wedge 0]}(1) \geq 0$.

The proof is included in [4], like the proof of the next theorem.

Theorem 2. For any $u \in \mathcal{H}_{BD}$ the next inequality is valid:

$$\delta_{u \vee (-u)}(1) \leq \delta_u(1).$$

Lemma 3. Let $u, v \in \mathcal{H}_{BD}$. Then for any $s \in S$, we have

$$\delta_{u \vee v}(s) + \delta_{u \wedge v}(s) \leq \delta_u(s) + \delta_v(s).$$

Proof. Using the Lemma 1, we get

$$\begin{aligned} \delta_u(s) + \delta_v(s) - 2\delta_{[u,v]}(s) &= \delta_{u-v}(s) = \delta_{(u-v) \vee 0 + (u-v) \wedge 0}(s) \geq \\ &\geq \delta_{(u-v) \vee 0}(s) + \delta_{(u-v) \wedge 0}(s) = \delta_{u \vee v - v}(s) + \delta_{u \wedge v - v}(s) = \\ &= \delta_{u \vee v}(s) + \delta_{u \wedge v}(s) + 2\delta_v(s) - 2\delta_{[v,v]}(s) - 2\delta_{[u \vee v, v]}(s) = \\ &= \delta_{u \vee v}(s) + \delta_{u \wedge v}(s) + 2\delta_v(s) - 2\delta_{[u+v, v]}(s) = \delta_{u \vee v}(s) + \delta_{u \wedge v}(s) - 2\delta_{[u, v]}(s). \end{aligned}$$

Proposition 4. If $u_n \in S$ is specifically increasing and such that there exists $M > 0$ and $A > 0$ for which

- i) $u_n \leq M \quad \forall n \in N$;
- ii) $\delta_{u_n}(1) \leq A \quad \forall n \in N$,

then if $u = \bigvee_{n \in N} u_n$, we have $\delta_u(1) < \infty$.

Proof. Let $v_n = u - u_n$. We observe that $0 \leq v_n \leq 2M$ and $\lim_{n \rightarrow \infty} v_n = 0$.

From [4], there exists $p_n \in S_P \cap S_B$ and $h_n \in S_H \cap S_B$, such that $v_n + p_n + h_n = 4Mv_n$. It results that $\lim_{n \rightarrow \infty} p_n = 0$ and p_n is dominated.

For any $p \in S_0$, $p \leq 1$, we have

$$\delta_{v_n}(p) = \frac{1}{2}\sigma(p_n)(p) = \frac{1}{2} \int_X p d\sigma(p_n) = \frac{1}{2} \int_X p_n d\sigma(p).$$

Thus we get $\lim_{n \rightarrow \infty} \delta_{u - u_n}(p) = 0$. Hence

$$\delta_u(p) = \lim_{n \rightarrow \infty} \delta_{v_n}(p) \leq \lim_{n \rightarrow \infty} \delta_{u_n}(1) \leq A.$$

Therefore

$$\sup_{p \in S_0, p \leq 1} \delta_u(p) = \delta_u(1) \leq A$$

and $u \in \mathcal{H}_{BD}$.

Proposition 4. *If $u \in \mathcal{H}_{BD}$, then the family $\{\delta_{u \wedge \alpha}(1)\}_{\alpha \in \mathbb{R}_+}$ is increasing.*

Proof. We apply Lemma 3. For any $\alpha, \beta > 0$, with $\alpha < \beta$, we get

$$\delta_{(u \wedge \beta) \wedge \alpha}(1) + \delta_{(u \wedge \beta) \vee \alpha}(1) \leq \delta_{u \wedge \beta}(1) + \delta_{\alpha}(1) = \delta_{u \wedge \beta}(1).$$

But $(u \wedge \beta) \wedge \alpha = u \wedge \alpha$ and from the above inequality we get

$$\delta_{u \wedge \alpha}(1) \leq \delta_{u \wedge \beta}(1).$$

Proposition 5. *If $u \in \mathcal{H}_{BD}$ then we have*

$$\lim_{\alpha \rightarrow \infty} \delta_{u \wedge \alpha}(1) = \delta_u(1).$$

Proof. Let $A_\alpha = [u \geq \alpha]$. Obviously, $\max(u, \alpha) = \alpha$ on $X \setminus A_\alpha$ and $\delta_{\max(u, \alpha)} = 0$ on $X \setminus A_\alpha$. Then we get

$$\delta_{u - \min(u, \alpha)} = \delta_{\max(u, \alpha)} = \chi_{A_\alpha} \delta_{\max(u, \alpha)}.$$

Hence

$$\delta_{u - \min(u, \alpha)}(1) = \delta_{\max(u, \alpha)}(1) \leq \delta_{\max(u, \alpha)}(A_\alpha) \rightarrow 0 \quad (\alpha \rightarrow \infty).$$

We deduce

$$(1) \quad \lim_{\alpha \rightarrow 0} \delta_{u - \min(u, \alpha)}(1) = 0.$$

For any $\alpha, \alpha' \in \mathbb{R}_+$, there exists $g_{\alpha\alpha'} \in [S_p \cap S_B]$ such that

$$\min(u, \alpha) - \min(u, \alpha') = u \wedge \alpha - u \wedge \alpha' + g_{\alpha\alpha'}.$$

and applying the properties of the gradient measure [4] we get

$$(2) \quad \delta_{u \wedge \alpha - u \wedge \alpha'}(1) + \delta_{g_{\alpha\alpha'}}(1) = \delta_{\min(u, \alpha) - \min(u, \alpha')}(1).$$

From the positivity of δ and the relation (1), we obtain

$$(3) \quad \delta_{\min(u, \alpha) - \min(u, \alpha')}^{1/2}(1) \leq \delta_{u - \min(u, \alpha)}^{1/2}(1) + \delta_{u - \min(u, \alpha')}^{1/2}(1), \quad \lim_{\alpha, \alpha' \rightarrow \infty} \delta_{u \wedge \alpha - u \wedge \alpha'}(1) = 0.$$

Again from [4], for any $p \in S_0$, with $p \leq 1$,

$$\delta_{u \wedge \alpha - u}(p) = \lim_{\alpha' \rightarrow \infty} \delta_{u \wedge \alpha - u \wedge \alpha'}(p) \leq \lim_{\alpha' \rightarrow \infty} \delta_{u \wedge \alpha - u \wedge \alpha'}(1).$$

Because the elements of S are increasing we get

$$\delta_{u \wedge \alpha - u}(1) \leq \lim_{\alpha' \rightarrow \infty} \delta_{u \wedge \alpha - u \wedge \alpha'}(1).$$

From the above inequality we have

$$\lim_{\alpha \rightarrow \infty} \delta_{u \wedge \alpha - u}(1) = 0 \quad \text{and} \quad \delta_u(1) = \lim_{\alpha \rightarrow \infty} \delta_{u \wedge \alpha}(1).$$

Theorem 6. *Let u be from S_B . The following assertions are equivalent:*

- i) $u \in \mathcal{H}_{BD}$;
- ii) $u = \bigvee_{\alpha \in \mathbb{R}_+} (u \wedge \alpha)$ and $\sup_{\alpha \in \mathbb{R}_+} \delta_{u \wedge \alpha}(1) < +\infty$.

Proof. Obviously, if $u \in S_H \cap S_B$ then $u = \bigvee_{\alpha \in \mathbf{R}_+} (u \wedge \alpha)$. If $u \in \mathcal{H}_{BD}$, from the previous propositions we have

$$\sup_{\alpha \in \mathbf{R}_+} \delta_{u \wedge \alpha}(1) = \delta_u(1) < \infty.$$

Conversely, if $\sup_{\alpha \in \mathbf{R}_+} \delta_{u \wedge \alpha}(1) < \infty$, then applying Proposition 4, we get $\delta_u(1) < \infty$ and $u \in \mathcal{H}_{BD}$.

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FUNCTII MĂRGINITE CU INTEGRALA DIRICHLET FINITĂ PE H-CONURI STANDARD

(Rezumat)

Se dă o caracterizare (Teorema 6) a elementelor substractibile mărginite care au integrala Dirichlet finită, într-un H -con standard de funcții. Cu ajutorul unor ipoteze suplimentare asupra H -conului standard, se poate defini o noțiune de integrală Dirichlet pentru elemente substractibile, mărginite, care să fie pozitivă [4]. Mulțimea elementelor mărginite, cu integrală Dirichlet finită $\mathcal{H}_{BD}$ este o latice față de ordinea specifică în care „contractia modul” operează (Teorema 2) și care este condițional completă (Propoziția 4).

QUELQUES OBSERVATIONS CONCERNANT LES DÉPLACEMENTS ADMISSIBLES

PAR

CĂTĂLINA DAVIDEANU

L'étude des opérateurs Γ et Γ^* introduits grâce aux notions de déplacement intérieur et de déplacement adhérent a suggéré la recherche de quelques problèmes analogues pour l'opérateur K introduit par la notion de déplacement admissible.

Dans ce qui suit on considère (X, τ) un espace linéaire topologique, $A \subset X$ un sous-ensemble quelconque de X et $x_0 \in X$.

Définition 1. On dit que $x \in X$ est un déplacement intérieur pour A à partir de $x_0 \in X$ s'il existe un voisinage V_x de x et un nombre $\varepsilon \in \mathbb{R}_+^*$ de sorte que pour tout $y \in V_x$ et tout $\eta \in]0, \varepsilon[$ on ait $x_0 + \eta y \in A$.

On notera $\Gamma(A; x_0)$ l'ensemble des déplacements intérieurs pour A à partir de x_0 . Il résulte directement, de la Définition 1, que $\Gamma(A; x_0)$ est un cône de sommet o .

Définition 2. On dit que $x \in X$ est un déplacement adhérent pour A à partir de $x_0 \in X$, si pour tout voisinage V_x de x et tout nombre $\varepsilon \in \mathbb{R}_+^*$, il existe $y \in V_x$ et $\eta \in]0, \varepsilon[$ de sorte qu'on a $x_0 + \eta y \in A$.

On notera $\Gamma^*(A; x_0)$ l'ensemble des déplacements adhérents pour A à partir de x_0 . Il résulte directement, de la Définition 2, que $\Gamma^*(A; x_0)$ est un cône de sommet o .

Définition 3. On dit que $x \in X$ est un déplacement admissible pour A à partir de $x_0 \in X$ s'il existe un nombre $\varepsilon \in \mathbb{R}_+^*$ tel que pour tout $\eta \in]0, \varepsilon[$ on ait $x_0 + \eta x \in A$.

On indique $K(A; x_0)$ l'ensemble des déplacements admissibles pour A à partir de x_0 . Il résulte aisément que $K(A; x_0)$ est un cône de sommet o .

Exemples. 1. Soit $X = \mathbb{R}^2$, $A = \{(x, y) \in \mathbb{R}^2 \mid x \geq 0 \text{ et } |y| \leq x^2\}$ et $z_0 = (0, 0)$. Alors

$$\Gamma(A; z_0) = \emptyset,$$

$$\Gamma^*(A; z_0) = \{(x, y) \in \mathbb{R}^2 \mid x \geq 0 \text{ et } y = 0\},$$

$$K(A; z_0) = \{(x, y) \in \mathbb{R}^2 \mid x \geq 0 \text{ et } y = 0\}.$$

Il est évident que, dans ce cas on a $\Gamma^*(A; z_0) = K(A; z_0)$.

Nous démontrerons seulement la relation pour $K(A; z_0)$.

Soit $z=(x, y)$ un point quelconque de $K(A; z_0)$ cela veut dire qu'il existe $\varepsilon \in \mathbb{R}_+^*$ de sorte que pour tout $\eta \in]0, \varepsilon[$ on ait $z_0 + \eta z \in A$.

Donc $(0, 0) + (\eta x, \eta y) \in A$ c'est-à-dire $\eta x \geq 0$ et $|\eta y| \leq \eta^2 x^2$. On obtient $x \geq 0$ et $|y| \leq \eta x^2$ pour tout $\eta \in]0, \varepsilon[$ donc on a $x \geq 0$ et $y=0$. On a aussi $\{(x, y) \in \mathbb{R}^2 \mid x \geq 0 \text{ et } y=0\} \subset K(A, z_0)$.

2. Soit $X = \mathbb{R}^2$, $A = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$ et $z_0 = (1, 0)$. Alors

$$\Gamma(A; z_0) = \{(x, y) \in \mathbb{R}^2 \mid x < 0\},$$

$$\Gamma^*(A; z_0) = \{(x, y) \in \mathbb{R}^2 \mid x \leq 0\},$$

$$K(A; z_0) = \{(x, y) \in \mathbb{R}^2 \mid x < 0\} \cup \{z_0\} = \Gamma(A; z_0) \cup \{z_0\}.$$

Soit $z=(x, y)$ un point quelconque de $K(A; z_0)$; cela signifie qu'il existe $\varepsilon \in \mathbb{R}_+^*$ de manière que pour tout $\eta \in]0, \varepsilon[$ on ait $z_0 + \eta z \in A$. Donc $(1, 0) + (\eta x, \eta y) \in A$, c'est-à-dire $(1 + \eta x, \eta y) \in A$.

En tenant compte de la définition de A on obtient

$$(1 + \eta x)^2 + \eta^2 y^2 \leq 1 \quad \text{ou} \quad x \leq -\frac{\eta}{2} (x^2 + y^2) \leq 0.$$

Parmi les éléments de la forme $(0, y) \in \mathbb{R}^2$ seulement $(0, 0) = z_0$ appartient à $K(A; z_0)$.

Soit z un point quelconque de l'ensemble $\{(x, y) \in \mathbb{R}^2 \mid x < 0\}$. Si on considère $\varepsilon = -2x/(x^2 + y^2)$ on a $z_0 + \eta z \in A$ pour chaque $\eta \in]0, \varepsilon[$.

3. Soit $X = \mathbb{R}^2$, $A = \left\{ \left(\frac{1}{n}, \frac{1}{n+1} \right) \in \mathbb{R}^2 \mid n \in \mathbb{N}^* \right\}$ et $z_0 = (0, 0)$. Alors

$$\Gamma(A; z_0) = \emptyset,$$

$$\Gamma^*(A; z_0) = \{(x, y) \in \mathbb{R}^2 \mid x \geq 0 \text{ et } y = x\},$$

$$K(A; z_0) \subset \{(x, y) \in \mathbb{R}^2 \mid x \geq 0 \text{ et } y = x\} = \Gamma^*(A; z_0).$$

Soit $z=(x, y)$ un point quelconque de $K(A; z_0)$; il existe alors $\varepsilon \in \mathbb{R}_+^*$ de sorte que pour tout $\eta \in]0, \varepsilon[$ on ait $z_0 + (\eta x, \eta y) \in A$, c'est-à-dire $\eta x = 1/n$ et $\eta y = 1/(n+1)$.

En éliminant n des deux relations ci-dessus on obtient $y = x/(1 + \eta x)$ pour tout $\eta \in]0, \varepsilon[$, ce qui démontre que $x = y$; en plus, on a $x = 1/(\eta n) \geq 0$. Donc on a $K \subset \{(x, y) \in \mathbb{R}^2 \mid x \geq 0 \text{ et } y = x\}$.

En utilisant les Définitions (1), (2) et (3) on peut établir des liaisons entre les trois cônes de déplacements dont nous avons parlé.

Proposition 1. Pour tout $A \subset X$ et $x_0 \in X$ on a les relations d'inclusion suivantes :

$$(a) \quad \Gamma(A; x_0) \subset \Gamma^*(A; x_0),$$

$$(b) \quad \Gamma(A; x_0) \subset K(A; x_0),$$

$$(c) \quad K^*(A; x_0) \subset \Gamma^*(A; x_0) \quad \text{où} \quad K^*(A; x_0) = cK(cA; x_0).$$

Démonstration. (a) Cette inclusion se déduit directement des Définitions (1) et (2).

(b) Soit $x \in \Gamma(A; x_0)$; d'après la Définition (1), il existe un voisinage V_x de x , et un nombre $\varepsilon \in \mathbb{R}_+^*$ de sorte que pour tout $y \in V_x$ et tout $\eta \in]0, \varepsilon[$ on ait $x_0 + \eta y \in A$. En particulier en prenant $y = x$ on obtient $x_0 + \eta x \in A$, donc $x \in K(A; x_0)$.

(c) À l'aide des Définitions (1) et (2) on peut prouver les relations suivantes :

$$\Gamma(A; x_0) = c\Gamma^*(cA; x_0), \quad \Gamma^*(A; x_0) = c\Gamma(cA; x_0).$$

En utilisant la relation (b) pour l'ensemble cA et tenant compte des relations ci-dessus et de la notation faite, on obtient (c).

Proposition 2. Si $A \subset B$ alors on a l'inclusion $K(A; x_0) \subset K(B; x_0)$ ce qui signifie que $K(\cdot, x_0) : \mathcal{P}(X) \rightarrow \mathcal{D}(X)$ est une application isotone.

Proposition 3. Soit $(A_i)_{i \in I}$ une famille de parties de X .

(a) On a toujours

$$(1) \quad K\left(\bigcap_{i \in I} A_i; x_0\right) \subset \bigcap_{i \in I} K(A_i; x_0),$$

$$(2) \quad K\left(\bigcup_{i \in I} A_i; x_0\right) \supset \bigcup_{i \in I} K(A_i; x_0).$$

(b) Si l'ensemble d'indices I est fini, on a

$$(3) \quad K\left(\bigcap_{i \in I} A_i; x_0\right) = \bigcap_{i \in I} K(A_i; x_0),$$

mais, en général, l'inclusion (2) est stricte.

(c) Si l'ensemble d'indices I n'est pas fini, l'égalité (3) n'est pas nécessairement vérifiée.

Proposition 4.

(a) $0 \in K(A; x_0)$ si et seulement si $x_0 \in A$;

b) $K(A; x_0) = K(A - x_0; 0)$;

(c) $K(A; x_0) \subset \bigcup_{\lambda > 0} \lambda(A - x_0)$.

Les propositions (2)–(4) sont des conséquences immédiates de la Définition (3).

Proposition 5. Soit $f : X \rightarrow \mathbb{R}$ une fonctionnelle linéaire, continue $f \neq 0$ et $\lambda \in \mathbb{R}$. On considère l'ensemble suivant $D = \{x \in X \mid f(x) < \lambda\}$; alors on a

$$K(D; x_0) = \begin{cases} X & \text{si } f(x_0) < \lambda, \\ D_0 & \text{si } f(x_0) = \lambda, \end{cases}$$

où $D_0 = \{x \in X \mid f(x) < 0\}$; $K(D; x_0) \subset D_0$ si $f(x_0) > \lambda$.

Démonstration. L'ensemble D est un convexe ouvert non vide.

Si $f(x_0) < \lambda$ alors $x_0 \in D = D$ donc $X = \Gamma(D; x_0) \subset K(D; x_0)$, c'est-à-dire $K(D; x_0) = X$.

Si $f(x) = \lambda$ alors, en considérant $x \in K(D; x_0)$ on a assurée l'existence d'un nombre $\varepsilon \in \mathbb{R}_+^*$ tel que $x_0 + \eta x \in D$ pour tout $\eta \in]0, \varepsilon[$.

On obtient $f(x_0) + \eta f(x) < \lambda$ donc $f(x) < 0$ d'où $x \in D_0$.

Si $f(x_0) > \lambda$, en procédant de la même manière on a $f(x_0) + \eta f(x) < \lambda$ ou $f(x) < \lambda - f(x_0) < 0$ d'où $f(x) < 0$.

Proposition 6. Si $A \subset X$ est convexe alors $K(A; x_0)$ est convexe pour tout $x_0 \in X$.

Démonstration. Soient $x_1, x_2 \in K(A; x_0)$ et $\lambda_1, \lambda_2 \in \mathbb{R}_+$ avec $\lambda_1 + \lambda_2 = 1$. En vertu de la Définition (3) on a assurée l'existence des éléments $\varepsilon_i \in \mathbb{R}_+$ tels que pour tout $\eta_i \in]0, \varepsilon_i[$ on ait $x_0 + \eta_i x_i \in A$ avec $i = 1, 2$.

Considérons $\varepsilon = \min(\varepsilon_1, \varepsilon_2)$; pour tout $\eta \in]0, \varepsilon[$ on obtient $x_0 + \eta x_i \in A$ où $i = 1, 2$. Du fait que A est convexe il résulte que $\lambda_1(x_0 + \eta x_1) + \lambda_2(x_0 + \eta x_2) \in A$ ou $(\lambda_1 + \lambda_2)x_0 + \eta(\lambda_1 x_1 + \lambda_2 x_2) \in A$ ce qui signifie que l'élément $\lambda_1 x_1 + \lambda_2 x_2 \in K(A; x_0)$, donc $K(A; x_0)$ est convexe.

On sait bien qu'en utilisant les opérateurs Γ et Γ^* on peut caractériser le minimum d'une fonctionnelle sur une intersection finie $\bigcap_{i \in I} A_i$, $(A_i)_{i \in I} \subset \mathcal{P}(X)$. De même, on peut établir dans le cadre indiqué une condition nécessaire. Lorsque les parties A_i , $i \in I$, et la fonctionnelle sont convexes et on impose quelques hypothèses supplémentaires, la condition dont nous avons parlé, est également suffisante.

Dans ce qui suit on va démontrer un théorème de même type pour le cône de déplacements admissibles.

Soient $A_1, A_2, \dots, A_n$ sous ensembles de X , $A = \bigcap_{i=1}^n A_i$ et $f: X \rightarrow \mathbb{R}$. On dit que $x \in A$ est un point de minimum, pour f , sur A , si on a $f(x) = \min_{x \in A} f(x)$.

Théorème 1. Si $f: X \rightarrow \mathbb{R}$ est minimum sur $A = \bigcap_{i=1}^n A_i$ en $\bar{x}$, alors en notant

$$\begin{aligned} A_0 &= \{x \in X \mid f(x) < f(\bar{x})\}, \\ \Omega_i &= K(A_i; \bar{x}) \text{ pour } i \in \{0, n-1\}, \\ \Omega_n &= K^*(A_n; \bar{x}), \end{aligned}$$

on a

$$\bigcap_{i=0}^n \Omega_i = \emptyset.$$

Démonstration. On montre que s'il existait $y \in \bigcap_{i=0}^n \Omega_i$ alors f ne serait pas minimum sur A en $\bar{x}$.

On suppose que $y \in \bigcap_{i=0}^n \Omega_i = (\bigcap_{i=0}^{n-1} \Omega_i) \cap \Omega_n$.

Comme $y \in \bigcap_{i=0}^{n-1} \Omega_i$, d'après la Proposition 3 on a $y \in K(\bigcap_{i=0}^{n-1} A_i, \bar{x})$ alors il existe $\varepsilon_0 \in \mathbb{R}_+$ tel que pour chaque $\eta \in]0, \varepsilon_0[$ on a

$$(*) \quad \bar{x} + \eta y \in \bigcap_{i=0}^{n-1} A_i.$$

Mais y appartient aussi à $\Omega_n = K(A_n; \bar{x}) = cK(cA_n; \bar{x})$: pour tout $\varepsilon \in \mathbb{R}_+$ il existe $\eta_\varepsilon \in]0, \varepsilon[$ de manière que $x + \eta_\varepsilon y \in cA_n$ ou

$$(**) \quad x + \eta_\varepsilon y \in A_n.$$

En particulier pour $\varepsilon_0 \in \mathbb{R}_+$ il existe $\eta_{\varepsilon_0} \in]0, \varepsilon_0[$ de sorte que $\bar{x} + \eta_{\varepsilon_0} y \in A_n$.

Puisque la relation (*) a lieu pour tout $\eta \in]0, \varepsilon_0[$ il résulte que $\bar{x} + \eta_{\varepsilon_0} y \in \bigcap_{i=0}^{n-1} A_i$.

Ainsi, finalement l'élément $z = \bar{x} + \eta_{\varepsilon_0} y \in \bigcap_{i=0}^* A_i$.

On a donc $z \in A$ et $z \in A_0(f(z) < f(\bar{x}))$ ce qui contredit le fait que f est minimum sur A en $\bar{x}$.

Le problème des conditions suffisantes pour qu'on ait un point minimum pour une fonctionnelle, reste encore ouvert.

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CÎTEVA OBSERVAȚII PRIVIND CONURILE DE DEPLASĂRI ADMISIBILE

(Rezumat)

Studiul operatorilor Γ și Γ^* introduși cu ajutorul noțiunilor de deplasare interioară și de deplasare admisibilă a sugerat cercetarea unor probleme analoage pentru operatorul K introdus prin intermediul noțiunii de deplasare admisibilă. Se obțin legături între cei trei operatori despre care am vorbit mai sus (Propoziția 1).

Folosind noțiunea de con de deplasări admisibile se stabilește o condiție necesară pentru ca o funcțională să-și atingă un minim pe o intersecție finită de părți ale unui spațiu linlar topologic (Teorema 1).

UNE DIVISION DU SIMPLEX EN POLYÈDRES AVEC APPLICATIONS DANS LA CHIMIE ET LA TECHNIQUE

PAR

DAVID RIMER et AL. CĂRĂUȘU

1. Introduction. Beaucoup des diagrammes d'équilibre des systèmes à $n+1$ composants qu'on rencontre dans la chimie, métallurgie etc. sont des représentations géométriques de certaines fonctions empiriques

$$(1.1) \quad f: \sigma^n \rightarrow \mathbb{R},$$

où σ^n est le simplex de l'espace affine $\mathcal{A}_n$, et f est une fonction de classe C^a ($a \geq 1$).

Ainsi, dans le cas le plus simple, $n=1$, donc si l'on considère tous les mélanges obtenus avec deux substances A_1 et A_0 , prises en proportions x et $1-x$, la fonction

$$(1.2) \quad t: [0, 1] \rightarrow \mathbb{R}, \quad t=t(x),$$

où t est la température de fonte du mélange, a les propriétés :

il-y-a E , $E \in (0, 1)$, tel que $t(E) < t(x)$, ($\forall x \in [0, 1] - \{E\}$);

$t'(x) < 0$ si $x \in [0, E]$, $t'(x) > 0$ si $x \in [E, 1]$,

$$(1.3) \quad \lim_{x \rightarrow E-0} \frac{dt}{dx} \neq \lim_{x \rightarrow E+0} \frac{dt}{dx}, \quad t''(x) < 0.$$

Dans ces circonstances, si, pour approximer f , on recourt à une fonction empirique de classe C^a ($a \geq 1$), définie par une seule expression analytique sur l'intervalle $[0, 1]$, les erreurs fournies par cette fonction sont grossières. Il est préférable alors de faire une division (Δ_1) du simplex σ^1 en deux intervalles $[0, E]$, $[E, 1]$ et à construire deux fonctions empiriques C^a ($a \geq 1$), définies sur ces deux intervalles; les erreurs fournies par ces deux fonctions sont plus petites. Vu que le mélange composé de deux substances A_1 et A_0 , prises dans les proportions E , respectivement $1-E$, est nommé dans la chimie *eutectique*, nous allons nommer cette division (Δ_1) de σ^1 , la *division eutectique* du simplex σ^1 . Remarquons aussi que : a) cette division est naturelle, vu qu'elle reflète certaines propriétés des mélanges binaires; b) la connaissance de cette division est une *condition nécessaire* (évidemment, pas suffisante) pour la construction des deux restrictions de la fonction (1.2).

La division analogue du simplex σ^2 peut être réalisée aisément sur un dessin. Pour le simplex σ^3 , on a réalisé une image de cette division à l'aide des anaglyphes [1], mais elle ne peut pas être utile pour la construction des restrictions C^a ($a \geq 1$) de la fonction (1.2). Si $n > 3$, le problème n'a pas reçu encore une solution.

Afin de combler cette lacune, nous avons entrepris cette étude.

2. La division eutectique du simplex σ^n . Soit $\sigma^n = [A_0, A_1, A_2, \dots, A_n]$ le simplex de l'espace affine $\mathcal{A}_n$, et $\sigma_{i_m}^m$ ($1 \leq m \leq n$; $1 \leq i_m \leq C_{n+1}^{m+1}$) ses faces de dimension m . Considérons, à l'intérieur de chaque face $\sigma_{i_m}^m$, un point fixé $E_{i_m}^m$. On peut réaliser une division unique de σ^n en $n+1$ polyèdres, en procédant comme il suit.

On construit les divisions (Δ_1) des faces $\sigma_{i_1}^1 \subset \sigma^n$ et l'on obtient, sur chaque arête $[A_\alpha, A_\beta] \subset \sigma^n$, deux segments $[A_\alpha, E_{i_1}^1]$, $[E_{i_1}^1, A_\beta]$. Puis on divise chaque face $\sigma_{i_2}^2 = [A_\alpha, A_\beta, A_\gamma] \subset \sigma^n$ en trois régions, plaques quadrilatères, en unissant par des segments de droite le point $E_{i_2}^2 \in \sigma_{i_2}^2$, avec les points $E_{i_1}^1$ des arêtes $[A_\alpha, A_\beta]$, $[A_\beta, A_\gamma]$, $[A_\gamma, A_\alpha]$. Notons ces divisions par (Δ_2) .

De la même manière le point $E_{i_3}^3$, intérieur à la face $\sigma_{i_3}^3 = [A_\alpha, A_\beta, A_\gamma, A_\delta] \subset \sigma^n$, uni avec les extrémités des segments $[E_{i_2}^2, E_{i_1}^1]$ qui ont réalisé les divisions (Δ_2) des faces $\sigma_{i_2}^2$, détermine 12 plaques triangulaires qui réalisent la division (Δ_3) de la face $\sigma_{i_3}^3$ en quatre régions. D'une manière analogue le processus continue de m à $m+1$ jusqu'à σ^n . Chaque division (Δ_m) étant unique il en résulte que la division (Δ_n) est unique.

Définition 2.1. La division (Δ_n) du simplex σ^n réalisée de la manière décrite ci-dessus sera nommée *division eutectique* du simplex σ^n .

Le nom est suggéré par l'origine du problème dans la chimie ou dans la métallurgie où les points $E_{i_m}^m$ représentent les eutectiques des systèmes à $m+1$ composants.

Les régions obtenues par une division eutectique sont, évidemment, des polyèdres. Si $n=1$ ou 2 ces polyèdres se réduisent à des segments, respectivement à des plaques quadrilatères planes. Parmi les sommets de chacun de ces polyèdres, l'un et seulement un est un sommet A_α du simplex σ^n . C'est pourquoi nous allons noter ces polyèdres par $[A_\alpha]$. Ces polyèdres ont la propriété que, si un point $M \in \sigma^n$ appartient à un polyèdre $[A_\alpha]$ alors tout point $M' \in \sigma^n$ tel que M appartient au segment orienté $[E_{i_1}^1, M']$ ou M' appartient au segment orienté $[E_{i_1}^1, M]$ appartient aussi au polyèdre $[A_\alpha]$; donc, les polyèdres $[A_\alpha]$ sont des parties de cônes, ayant le sommet commun $E_{i_1}^1$.

La frontière d'un tel polyèdre est formée de *faces extérieures* contenues dans les faces σ^{n-1} et σ^n et de *faces intérieures* situées à l'intérieur de σ^n . Toutes ces faces sont des portions de $n-1$ -plans; en particulier, les faces intérieures sont des simplex σ^{n-1} .

Proposition 2.1. Chaque polyèdre $[A_\alpha]$ de σ^n a $V=2^n$ sommets, $F_e=n$ faces extérieures et $F_i=n!$ faces intérieures.

En effet par un sommet arbitraire A_α de σ^n passent C_n^m faces $\sigma^m \subset \sigma^n$ et à l'intérieur de chacune de ces faces il y a un point fixé, $E_{i_m}^m$. Alors le

nombre de tous les points $E_{i_m}^m$ est $\sum_{m=1}^n C_n^m = 2^n - 1$. Puisque ces points

$E_{i_m}^m$ et le point A_α constituent l'ensemble de tous les sommets du polyèdre $[A_\alpha]$ il en résulte que $V=2^n$.

Si $P^m (1 \leq m \leq n)$ sont les polyèdres résultés de la division (Δ_m) d'une face $\sigma^m \subset \sigma^n$ on obtient par induction, les relations de récurrence

$$(2.1) \quad F_e(P^{m+1}) = F_e(P^m) + 1, \quad F_i(P^{m+1}) = (m+1)F_i(P^m) \quad (1 \leq m \leq n-1).$$

En donnant à m les valeurs 1, 2, ..., $n-1$, vu que $F_e(P^1) = F_i(P^1) = 1$, on obtient $F_e(P^n) = n$ et $F_i(P^n) = n!$.

3. Relations entre une division eutectique et une division simplicielle d'un simplexe σ^n . 3.1. Soit (Δ) une division eutectique et (D) — une division simplicielle d'un même simplexe σ^n . Nous nous proposons de préciser, pour chaque noeud de la division (D) , le polyèdre $[A_\alpha]$ auquel il appartient. Ce problème s'impose, non seulement par des raisons théoriques, mais aussi par son aspect pratique, dans le cas où $E_{i_m}^m$ sont effectivement des eutectiques. En effet, en identifiant un polyèdre $[A_\alpha]$ avec l'ensemble des noeuds de (D) qu'il contient, vu que pour tous les noeuds P de (D) on peut connaître, expérimentalement la température de fonte des mélanges P , il devient possible à déterminer des fonctions empiriques de classe $C^a (a \geq 1)$ et donc $t: [A_\alpha] \rightarrow \mathbb{R}$, $t = t(P)$.

Pour $n=2$, la division est obtenue directement, à l'aide du graphique (fig. 1). Nous allons nous occuper du cas $n=3$, les cas $n > 3$ pouvant être résolus d'une manière analogue.

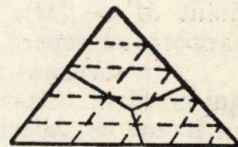


Fig. 1

Considérons les sommets A_α et les points $E_{i_m}^m (m=1, 2, 3; i_m=1, \dots, 6 \text{ si } m=1, i_m=1, \dots, 4 \text{ si } m=2, i_m=1 \text{ si } m=3)$. Ces 11 points $E_{i_m}^m$ déterminent la division eutectique (Δ_3) . Il résultent 4 régions, les polyèdres $[A_\alpha] \alpha=0, 3$, ayant un sommet commun, E_1^3 . Les faces intérieures des polyèdres $[A_\alpha]$ sont des plaques triangulaires $\pi_{i_1, i_2} = [E_1^3, E_{i_2}^2, E_{i_1}^1]$, où $[E_{i_2}^2, E_{i_1}^1]$ sont les segments qui constituent la frontière intérieure des régions planes — faces extérieures de $[A_\alpha]$.

L'application $r: (D) \rightarrow (\Delta)$ qui associe à chaque noeud $P \in (D)$ un polyèdre $[A_\alpha] \subset (\Delta)$ est multivoque, parce-qu'un noeud P peut appartenir à un, deux, trois ou même à tous les quatre polyèdres $[A_\alpha]$. Nous allons réaliser effectivement l'application $r: (D) \rightarrow (\Delta)$ de deux manières: 1) par une construction graphique; 2) par des considérations analytiques et à l'aide d'un ordinateur.

3.2. Solution graphique. Puisqu'il n'est pas possible de réaliser une application bijective et continue de $\sigma^3 \subset \mathbb{R}^3$ en $\mathbb{R}^2$, nous considérons un σ^3 „raréfié“ nommé „pseudotétraèdre“ [2]

$$(3.1) \quad s^3 = \sigma^3 \cap X_3^2, \quad \text{où } X_3^2 = \mathbb{Q}^2 \times \mathbb{R},$$

$\mathbb{Q}$ et $\mathbb{R}$ étant les ensembles des nombres rationnels, respectivement réels. $(X_3^2; +)$ est groupe abélien; l'opération externe $\mathbb{R} \times X_3^2 \rightarrow X_3^2$ est partielle, car si $\lambda \in \mathbb{R} - \mathbb{Q}$, alors $\lambda x \notin X_3^2$. $(X_3^2; +, \cdot)$ a été nommé $\mathbb{R}$ -espace pseudo-linéaire de type 1 [4]. Nous considérons cet espace muni de la topologie

induite par la topologie canonique de $\mathbb{R}^3$. Alors on peut définir une application continue (dont le cas général est étudié dans [5]),

$$(3.2) \quad \varphi : X_3^2 \rightarrow \mathbb{R}^2,$$

$$(3.2') \quad y^1 = ax^1 + x^2, \quad y^2 = bx^1 + x^3, \quad a \in \mathbb{R} - \mathbb{Q}, \quad a^2 + b^2 = 1.$$

Cette application est injective, car de $ax_1^1 + x_1^2 = ax_2^1 + x_2^2$, $a \in \mathbb{R} - \mathbb{Q}$, $x_1^1, x_1^2 \in \mathbb{Q}$ il résulte $x_1^1 = x_2^1$; ce qui implique aussi $x_1^2 = x_2^2$. Alors la restriction de (3.2)

$$(3.3) \quad \varphi_1 : X_3^2 \rightarrow \varphi(X_3^2) \subset \mathbb{R}^2$$

avec (3.2') est bijective et l'on a l'application inverse

$$\varphi_1^{-1} : \varphi(X_3^2) \rightarrow X_3^2.$$

En rapportant l'espace X_3^2 à un repère affine $R = \{A_0; A_1, A_2, A_3\}$ on obtient par (3.2') $R' = \varphi_1(R) = \{A'_0; A'_1, A'_2, A'_3\}$. Vu que (3.2) est une application affine, elle conserve le rapport simple de trois points collinéaires. Les coordonnées d'un point $M \in X_3^2$ étant définies à l'aide de certains rapports simples, il suit que l'on peut rapporter les points $M' = \varphi_1(M)$ à R' considéré comme un „repère“ de l'espace $\varphi_1(X_3^2)$ et attribuer, à chaque point $M' = \varphi_1(M)$, rapporté à R' , les coordonnées du point $M \in X_3^2$ par rapport au repère R .

En revenant au pseudotétraèdre s^3 on réalise dans un plan un dessin qui contient certains éléments essentiels de $\varphi_1(s^3)$, où φ_1 est particularisé en fixant les valeurs des coefficients a et b de (3.2'). Sur les droites A_0A_j ($j=1, 2, 3$), de même que sur les parallèles à elles, on marque les points dont les coordonnées, par rapport au repère R' , satisfont aux conditions $x^i \in \{(0, 1)t : t=0, 9, i=1, 2, 3, x^1 + x^2 + x^3 \leq 1\}$, en obtenant ainsi la division $(D') = \varphi_1(D)$. Tout de même on représente les plaques $\pi'_{i,t_1} = \varphi_1(\pi_{i,t_1})$ et $p'_{i,t_1} = \varphi(p_{i,t_1})$, où p_{i,t_1} sont les projections des plaques π_{i,t_1} parallèles à A_0A_3 sur le plan $A_0A_1A_2$. On obtient ainsi la division $(\Delta') = \varphi_1(\Delta)$.

Maintenant on peut réaliser effectivement l'application $\nu : (D') \rightarrow (\Delta')$.

a) Pour chaque noeud $M'(x_0^1, x_0^2, 0) \in (D')$, situé sur une plaque p'_{i,t_1} , on détermine, par une construction graphique simple, le point $M''(x_0^1, x_0^2, x^3)$, où la parallèle $(d) \quad A'_0A'_3$, menée par M' , coupe la plaque π'_{i,t_1} . Ces points M'' décomposent l'ensemble des noeuds de (D') , situés sur la parallèle (d) respective de manière que : (i) les noeuds appartenant à l'intérieur d'un même segment appartiennent au même polyèdre $[A_\alpha]$; (ii) les noeuds appartenant à des segments distincts de (d) appartiennent à des polyèdres $[A_\alpha]$ distincts; (iii) un noeud appartenant à une plaque π'_{i,t_1} , appartient aux deux ou trois polyèdres $[A_\alpha]$ qui contiennent dans leur frontière intérieure cette plaque ou sa frontière.

b) Pour les noeuds $M'(x_0^1, x_0^2, 0)$ qui n'appartiennent à aucune plaque p'_{i,t_1} , situés donc dans la proximité d'un sommet A_α de σ^3 , la parallèle (d) à $A'_0A'_3$ ne coupe aucune plaque π'_{i,t_1} ; tous les points de ces parallèles appartiennent au polyèdre $[A_\alpha]$.

3.2. Application. Supposons que, par rapport au repère R , les points A_α et $E_{i,m}^m$ ont les coordonnées données dans le Tableau 1.

Tableau 1

	A_0	A_1	A_2	A_3	E_1^1	E_2^1	E_3^1	E_4^1	E_5^1	E_6^1	E_1^2	E_2^2	E_3^2	E_4^2	E_5^1
x^1	0	1	0	0	0,6	0	0	0,7	0	0,5	0	0,5	0,4	0,3	0,2
x^2	0	0	1	0	0	0,5	0	0,3	0,8	0	0,4	0	0,3	0,3	0,2
x^3	0	0	0	1	0	0	0,6	0	0,2	0,5	0,5	0,3	0	0,4	0,5

Avec ces données, on peut conclure d'un dessin (un tétraèdre sur lequel sont représentés les points $E_{i_m}^m$) le polyèdre $[A_\alpha]$ auquel appartient — comme partie de la frontière intérieure — chaque plaque π_i, π_{i_1} . Le résultat est donné par la matrice booléenne du Tableau 2 (1 indique l'appartenance, 0 — la non-appartenance).

Tableau 2

	π_{12}	π_{13}	π_{15}	π_{21}	π_{23}	π_{26}	π_{31}	π_{32}	π_{34}	π_{44}	π_{45}	π_{46}
$[A_0]$	1	1	0	1	1	0	1	1	0	0	0	0
$[A_1]$	0	0	0	1	0	1	1	0	1	1	0	1
$[A_2]$	1	0	1	0	0	0	0	1	1	1	1	0
$[A_3]$	0	1	1	0	1	1	0	0	0	0	1	1

Dans la fig. 2, nous avons représenté $\varphi_1(s^3)$, où φ_1 est particularisé par $a = -\sqrt{13}/4$ et $b = -\sqrt{3}/4$; de même sur ce dessin sont tracées les parallèles à A_0A_2 , celles à A_0A_3 et trois des 12 plaques $\pi_{i,j}$, et leurs projections $p_{i,j}$. Par l'un des nombres 0, 1, 2, 3, écrit à côté de certains noeuds de la division (D'), nous avons indiqué que le noeud respectif appartient au polyèdre $[A_0]$, respectivement $[A_1]$, $[A_2]$, $[A_3]$. Tous les noeuds de (D') situés sur une parallèle à A_0A_3 , entre deux noeuds marqués par le même nombre α ($\alpha = 0, 3$), appartiennent au même polyèdre $[A_\alpha]$.

3.3. Solution analytique. Plus complexe et laborieuse que la solution graphique, la solution analytique a les étapes suivantes :

(i) On précise les faces intérieures et extérieures des polyèdres $[A_\alpha]$ et on écrit la caractérisation analytique de chacune de ces faces ; on adopte un ordre des plaques $\pi_{i_1} \dots \pi_{i_{n-1}}$, ordre d'ailleurs arbitraire.

(ii) Pour chacun des polyèdres $[A_\alpha]$, on établit s'il est convexe ou concave en intersectant le plan-soutien de chaque face intérieure $\pi_{i_1, i_2, \dots, i_{n-1}}$ de $[A_\alpha]$ avec ses faces extérieures, non-adjacentes à la plaque considérée. Si au moins un des polyèdres $[A_\alpha]$ est concave, il est décomposé — par ces intersections — en polyèdres convexes $[A_{\alpha,k}]$, $k=1, 2, \dots$ de sommet commun E_1^n . Afin d'uniformiser la notation, les polyèdres convexes $[A_\alpha]$ seront notés par $[A_{\alpha,1}]$. Les faces des polyèdres convexes $[A_{\alpha,k}]$ qui sont parties des faces extérieures de $[A_\alpha]$ seront nommées „faces extérieures“ de $[A_{\alpha,k}]$. Ce raffinement de la division (Δ_n) sera noté ($\tilde{\Delta}_n$).

(iii—1) Si $\pi_{i_1} \dots \pi_{i_{n-1}}(x^1, \dots, x^n) = 0$ est l'équation du $n-1$ -plan-soutien de la plaque $\pi_{i_1, \dots, i_{n-1}}$ et a été déterminée dans l'étape (i), pour chacune des faces intérieures $\pi_{i_1, \dots, i_{n-1}}$ de chaque polyèdre $[A_\alpha]$ on calcule

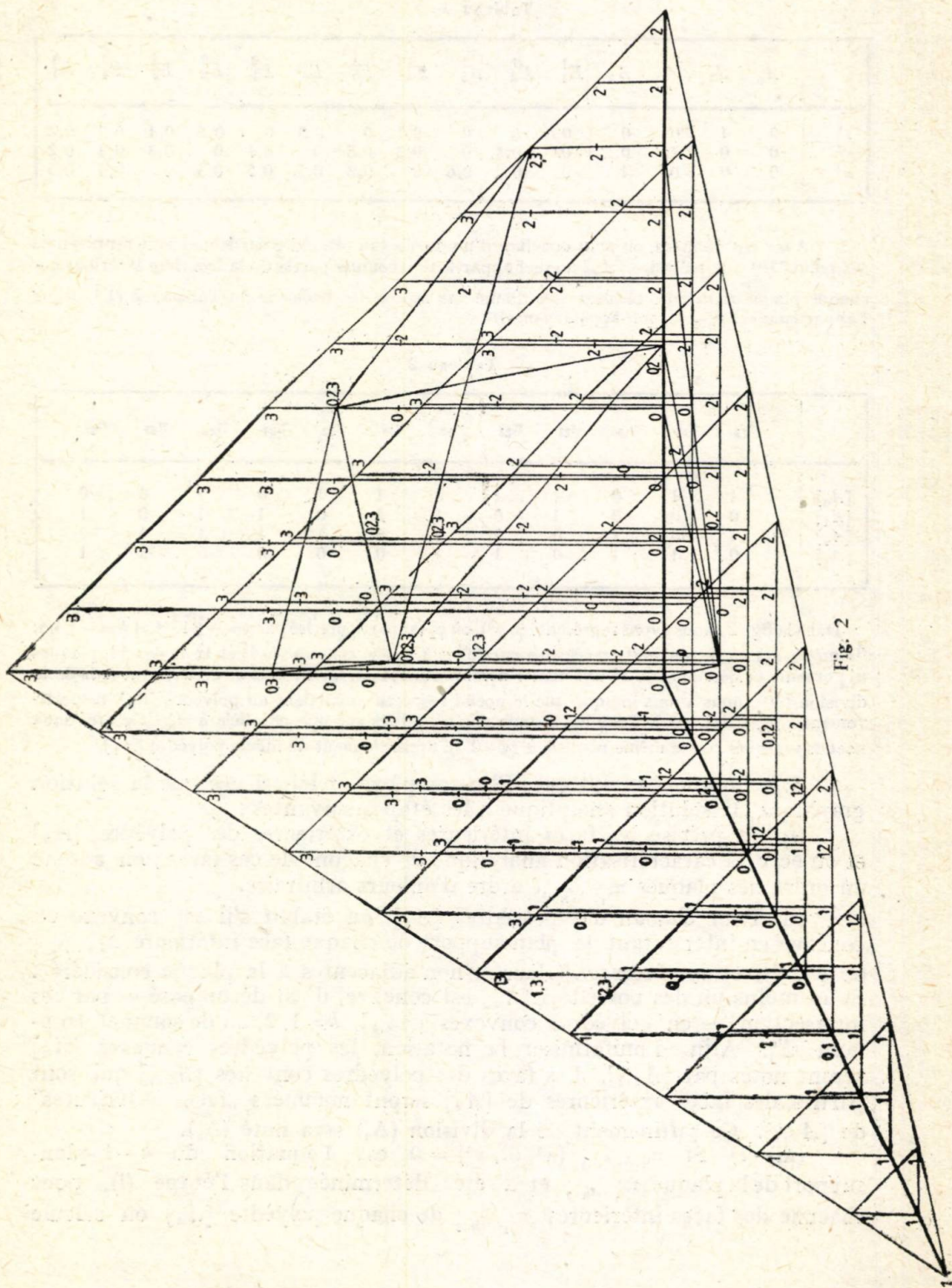


Fig. 2

$\text{sgn } \pi_{i_1, \dots, i_{n-1}}(A_\alpha)$. Puis, en considérant à l'intérieur de chacun des polyèdres $[A_{\alpha, k}]$, ou sur chacune de ses faces extérieures, un point arbitraire fixé M_0 , on détermine $\text{sgn } \pi_{i_1, \dots, i_{n-1}}(M_0)$ qui est égal à $\pm \text{sgn } \pi_{i_1, \dots, i_{n-1}}(A_\alpha)$ ou à zéro, selon la position relative de M_0 par rapport à A_α , respectivement si M_0 est situé sur le prolongement de la plaque $\pi_{i_1, \dots, i_{n-1}}$. On obtient ainsi, pour chaque point considéré M_0 , un système ordonné de $n!$ nombres, tous ces nombres étant $-1, 0$ ou 1 .

(iii-2) Il en résulte que un noeud M de la division simplicielle (D) de σ^n appartient à l'intérieur ou à la frontière extérieure d'un polyèdre $[A_\alpha]$ si et seulement si, pour toutes les faces intérieures $\pi_{i_1, \dots, i_{n-1}}$ de $[A_\alpha]$, $\text{sgn } \pi_{i_1, \dots, i_{n-1}}(M)$ coïncide avec l'un des systèmes ordonnés $\pi_{i_1, \dots, i_{n-1}}(M_0)$ obtenus à (iii-1).

(iv-1) Pour chaque plaque $\pi_{i_1, \dots, i_{n-1}}$, face intérieure d'un polyèdre $[A_\alpha]$, on considère la pyramide de sommet A_α et de base la plaque $[E_1^1, E_1^2, \dots, E_1^{n-1}, E_1^n] = \pi_{i_1, \dots, i_{n-1}}$. On écrit les équations des $n-1$ -plans-support des faces latérales de ces pyramides : soient-elles $F_j(x^1, \dots, x^n) = 0, j = 1, n$. Pour chacun des polynômes F_j , on calcule $\text{sgn } F_j(E_i^s)$, où E_i^s est le sommet de la plaque $\pi_{i_1, \dots, i_{n-1}}$ qui n'est pas contenu dans la face F_j .

(iv-2) Il en résulte que un noeud M de la division simplicielle (D) appartient à l'intérieur, respectivement à la frontière de la plaque $\pi_{i_1, \dots, i_{n-1}}$, si et seulement si l'on a $\text{sgn } F_j(E_i^s) \text{sgn } F_j(M) > 0, j = 1, n$, respectivement si $F_j(M) = 0 (j = 1 \vee 2 \vee \dots \vee n), \pi_{i_1, \dots, i_{n-1}}(M) = 0$ et chacune des coordonnées de M appartient à l'un des intervalles déterminés par les coordonnées du même nom des extrémités des arêtes respectives de la plaque $\pi_{i_1, \dots, i_{n-1}}$.

Si le noeud $M \in (D)$ appartient à une plaque $\pi_{i_1, \dots, i_{n-1}}$, alors il appartient aux deux polyèdres $[A_\alpha]$, pour lesquels cette plaque est face intérieure. Si M appartient à la frontière commune de deux plaques $\pi_{i_1, \dots, i_{n-1}}$, alors M appartient aux trois polyèdres $[A_\alpha]$, pour lesquels au moins l'une de ces plaques est face intérieure commune.

Évidemment, cette solution analytique implique un volume trop grand de calculs. À cette cause on peut s'aider, complètement ou au moins en partie, d'un ordinateur. Ainsi on peut introduire, comme données d'entrée du programme, les coordonnées de E_i^m aussi que les résultats obtenus dans les étapes (i), (ii), (iii-1) et (iv-1), en restant à l'ordinateur la charge

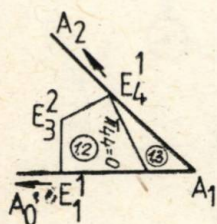


Fig. 3

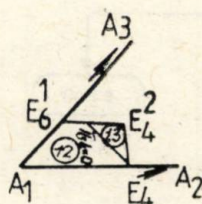


Fig. 4

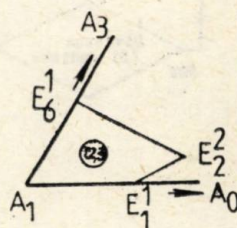


Fig. 5

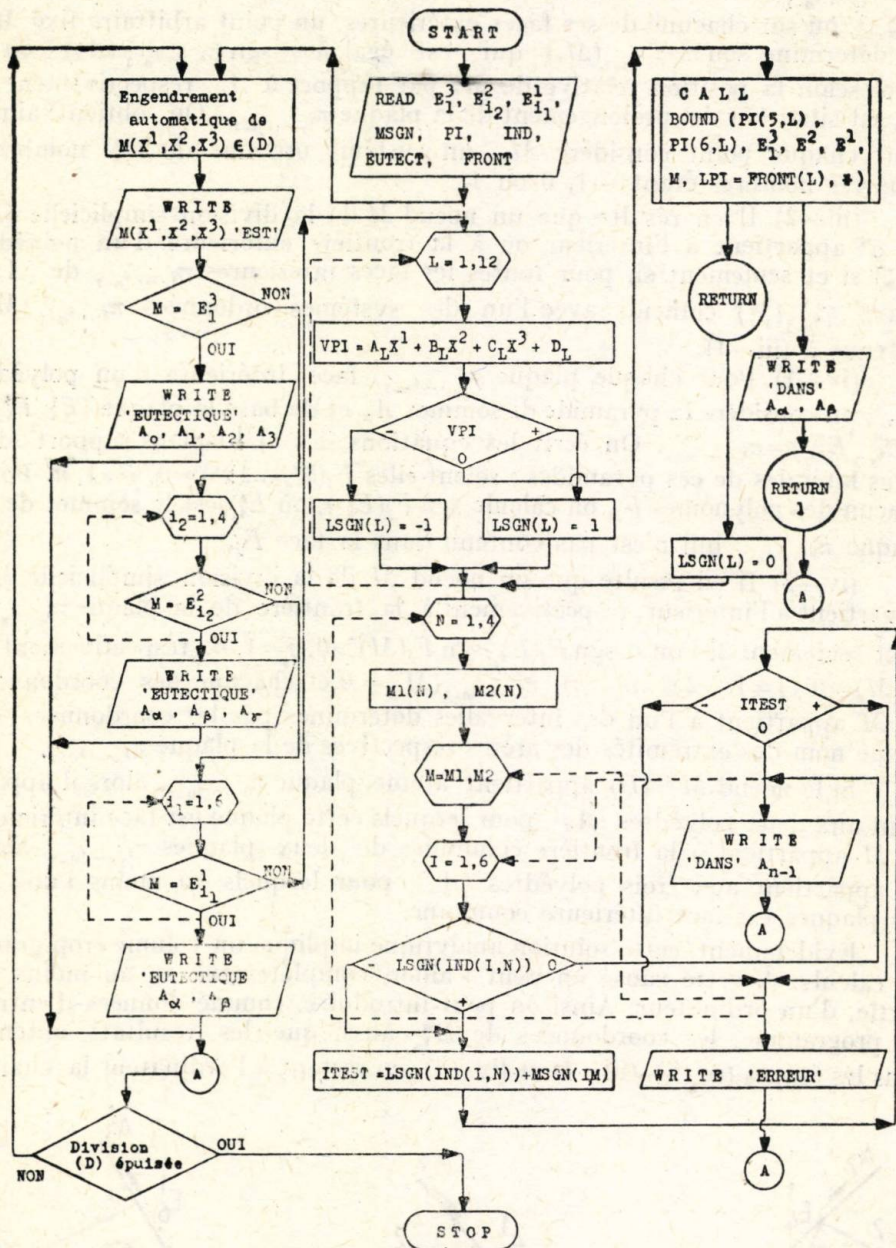


Fig. 6

d'engendrer les coordonnées des noeuds de la division simplicielle (D), puis, pour chaque noeud M , de vérifier si la condition prévue à (iii-2) est satisfaite et — dans le cas d'une réponse négative — laquelle des conditions prévues à (iv-2) est satisfaite.

Comme une application, nous avons repris l'exemple du § 3.2. Après avoir amplifié par 10 les coordonnées des points A_α et $E_{i_m}^m$ (pour des raisons de commodité), nous avons parcouru les étapes (i), (ii), (iii-1) et (iv-1). À (ii) il a résulté que tous les quatre polyèdres sont concaves, ce qui a impliqué la division ($\tilde{\Delta}_3$) de σ^3 en 32 polyèdres convexes $[A_{\alpha,k}]$ (7, 3, 3 ou 19 selon que $\alpha=0, 1, 2$ ou 3). Dans les figs. 3-5 nous donnons les divisions des faces extérieures de $[A_1]$, résultées par ($\tilde{\Delta}_3$), les nombres 1, 2 et 3 indiquant que les régions respectives appartiennent à $[A_{1,1}]$, $[A_{1,2}]$ ou $[A_{1,3}]$. À (iii-1), nous avons obtenu les séquences de $3! = 6$ nombres -1, +1 ou 0 (Tableau 3). Puis nous avons élaboré un programme FORTRAN (appelé CONDIV). Les données d'entrée de ce programme sont les coordonnées des points $E_{i_m}^m$ et les divers résultats partiels obtenus dans les étapes (i), (ii), (iii-1) et (iv-1). L'ordinateur a engendré la division simplicielle (D) et, pour chaque noeud $M \in (D)$, a testé les conditions prévues à (iii-2) et éventuellement à (iv-2).

Tableau 3

$[A_{1,k}] \backslash \pi_{i_1 i_2}$	π_{21}	π_{26}	π_{31}	π_{34}	π_{44}	π_{46}
$[A_{1,1}]$	1	1	1	-1	1	-1
$[A_{1,2}]$	1	1	1	-1	-1	-1
$[A_{1,3}]$	1	1	1	1	1	-1
$[A_{1,1}] \cap [A_{1,2}]$	1	1	1	-1	0	-1
$[A_{1,1}] \cap [A_{1,3}]$	1	1	1	0	1	-1

Mentionnons que la solution graphique et la solution analytique (à l'aide d'ordinateur) ont réalisé exactement la même application $r: (D) \rightarrow (\Delta_3)$, ce qui prouve l'avantage de la solution graphique (au moins pour $n=3$) puisqu'elle est plus simple. L'organigramme du programme CONDIV est donnée dans la fig. 6.

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O DIVIZIUNE A SIMPLEXULUI ÎN POLIEDRE CU APLICAȚII ÎN CHIMIE ȘI TEHNICĂ

(Rezumat)

Plecind de la o problemă studiată în [3], se introduce și se studiază „diviziunea eutectică” a simplexului σ^n al spațiului afin n -dimensional. Se determină, ca funcții de n , numărul de vîrfuri, fețe interioare și fețe exterioare ale poliedrelor rezultate din această diviziune. Se cercetează relațiile între diviziunea eutectică și diviziunea simplicială clasică a lui σ^n și se realizează efectiv corespondența utilizînd, fie o metodă grafică, fie una analitică. Pentru cazul $n=3$ se consideră un exemplu și se obține pe ambele căi exact același rezultat.

FAMILIES OF QUASI-CONNECTIONS ON DIFFERENTIABLE MANIFOLDS (I)

BY

ELENA PUȘCAȘU

1. Introduction. Let M be a C^∞ -differentiable, n -dimensional manifold and let be defined on M a nondegenerate tensor field F_j^i of type $(1, 1)$. Since the tensor field F_j^i is nondegenerate, there exists a tensor field G_j^i of type $(1, 1)$ which is given by the converse matrix of the tensor field F_j^i .

Following the way of [2], [3], we can introduce the notion of quasi-connection on M . If we consider the quasi-connection Γ_{jk}^i on M , every other quasi-connection $\tilde{\Gamma}_{jk}^i$ on M is given by the formula

$$(1) \quad \tilde{\Gamma}_{jk}^i = \Gamma_{jk}^i + X_{jk}^i,$$

where X_{jk}^i is an arbitrary tensor field of type $(1, 2)$. In the present paper we determine a family of quasi-connections on M and study some algebraic and geometric properties of these quasi-connections.

2. The Operators P and Q on M . We introduce the following tensor

$$(2) \quad P_{km}^{ih} = \frac{1}{n} F_m^i G_k^h, \quad Q_{km}^{ih} = \frac{1}{n} (n \delta_k^i \delta_m^h - F_m^i G_k^h).$$

Then, we consider the operations

$$(3) \quad \begin{cases} (PV)_{mh}^i = P_{kh}^{ij} V_{mj}^k, & (QV)_{mh}^i = Q_{kh}^{ij} V_{mj}^k, \\ (PH)_{mh}^{ia} = P_{gh}^{ij} H_{mj}^{ga}, & (QH)_{mh}^{ia} = Q_{gh}^{ij} H_{mj}^{ga}, \end{cases}$$

V and H being tensor fields of type $(1, 2)$ and $(2, 2)$ respectively.

It can be seen that the following relations hold:

$$(4) \quad P+Q=I \otimes I, \quad PP=P, \quad QQ=Q, \quad PQ=QP=0,$$

where I is the identity tensor field.

Hence P and Q are operators of Obata type [1].

3. Family of Quasi-Connections on M . We, now consider the tensor field V_{jk}^i defined by

$$(5) \quad V_{jk}^i = n F_m^i \nabla_k F_j^m - F_k^i \nabla_m F_j^m,$$

where ∇ is the operator of covariant derivative with respect to $\hat{\Gamma}$ [2]. It can be verified that

$$(6) \quad (PV)_{jk}^i = 0.$$

Now, we are interested in finding of the quasi-connections Γ given by (1) when the tensor field X satisfies the equation

$$(7) \quad (QX)_{jk}^i = V_{jk}^i.$$

The Eq. (7) has a solution if and only if $(PV)_{jk}^i = 0$, and then the general solution of this equation is given by

$$(8) \quad X_{jk}^i = V_{jk}^i + W_{jk}^i,$$

where W is an arbitrary tensor field of type (1, 2) that satisfies

$$(9) \quad (QW)_{jk}^i = 0.$$

Hence the general family of quasi-connections Γ is given by

$$(10) \quad \Gamma_{jk}^i = \hat{\Gamma}_{jk}^i + n F_m^i \nabla_k F_j^m - F_k^i \nabla_m F_j^m + W_{jk}^i.$$

Let $\Gamma, \bar{\Gamma}$ be two quasi-connections of the family (10).

For $\bar{\Gamma}$ we have

$$(11) \quad \bar{\Gamma}_{jk}^i = \hat{\Gamma}_{jk}^i + n F_m^i \nabla_k F_j^m - F_k^i \nabla_m F_j^m + \bar{W}_{jk}^i.$$

From (10) and (11) it follows

$$(12) \quad \bar{\Gamma}_{jk}^i = \Gamma_{jk}^i + S_{jk}^i,$$

where $S_{jk}^i = \bar{W}_{jk}^i - W_{jk}^i$ is an arbitrary tensor field which satisfies condition

$$(13) \quad (QS)_{jk}^i = 0.$$

Now, if we consider the transformation: $\Gamma \rightarrow \bar{\Gamma}$ defined by (12), with S satisfying (13), the set of these transformations is a group which is isomorphic to the additive group of tensors S .

Next we take

$$(14) \quad S_{jk}^i = F_k^i V_j,$$

with V_j an arbitrary covector field.

It easily follows that S defined by (14) satisfies (13), and hence we have

$$(15) \quad \bar{\Gamma}_{jk}^i = \Gamma_{jk}^i + F_k^i V_j.$$

The set of these transformations defines the group G .

4. The Invariants of the Group G . If we denote by $D, \bar{D}$ the operators of covariant derivative with respect to $\Gamma, \bar{\Gamma}$ respectively, then, by an usual computation we obtain the invariants

$$(16) \quad \begin{cases} \bar{D}_k F_j^i = D_k F_j^i, & \bar{D}_k G_j^i = D_k G_j^i, \\ \bar{D}_k P_{mj}^{ih} = D_k P_{mj}^{ih}, & \bar{D}_k Q_{mj}^{ih} = D_k Q_{mj}^{ih}. \end{cases}$$

Let $T_{jk}^i, \bar{T}_{jk}^i$ be the torsion tensor fields and $R_{hij}^k, \bar{R}_{hij}^k$ the curvature tensor fields of the quasi-connections $\Gamma, \bar{\Gamma}$ respectively.

From [2] and [15] we have

$$(17) \quad \bar{T}_{jk}^i = T_{jk}^i + F_k^i V_j - F_j^i V_k,$$

$$(18) \quad \bar{R}_{hij}^k = R_{hij}^k + (D_i F_h^k) V_j - (D_j F_h^k) V_i + F_h^k (D_i V_j - D_j V_i + T_{ij}^m V_m).$$

From (3) and (17) we obtain

$$(19) \quad (P\bar{T})_{kj}^i = (PT)_{kj}^i + \frac{n-1}{n} F_j^i V_k.$$

It follows

$$(20) \quad F_j^i V_k = \frac{n}{n-1} [(P\bar{T})_{kj}^i - (PT)_{kj}^i].$$

Substituting (20) in (17) we have

$$\bar{T}_{jk}^i = T_{jk}^i + \frac{n}{n-1} [(P\bar{T})_{jk}^i - (PT)_{jk}^i - (P\bar{T})_{kj}^i + (PT)_{kj}^i].$$

Thus we have obtained the invariant tensor field of the group G given by

$$(21) \quad I_{jk}^i = T_{jk}^i + \frac{n}{n-1} [(PT)_{kj}^i - (PT)_{jk}^i].$$

From (16), (17), (18) and (20) we can obtain, by some easy computations, the formula

$$(22) \quad \begin{aligned} \bar{R}_{hij}^k &= R_{hij}^k + \frac{n}{n-1} [\bar{D}_i (P\bar{T})_{jh}^k - \bar{D}_j (P\bar{T})_{ih}^k + \bar{T}_{ij}^m (P\bar{T})_{mh}^k] - \\ &- \frac{n}{n-1} [D_i (PT)_{jh}^k - D_j (PT)_{ih}^k + T_{ij}^m (PT)_{mh}^k]. \end{aligned}$$

Thus, we find another invariant tensor field of the group G , given by

$$(23) \quad J_{hij}^k = R_{hij}^k - \frac{n}{n-1} [D_i (PT)_{jh}^k - D_j (PT)_{ih}^k + T_{ij}^m (PT)_{mh}^k].$$

Now, we shall give a geometric interpretation for these invariants.

Let $\hat{\Gamma}$ be a quasi-connection of the group G given by

$$(24) \quad \hat{\Gamma}_{jk}^i = \Gamma_{jk}^i + F_k^i \hat{V}_j.$$

From (17) and (24) we have

$$(25) \quad \hat{T}_{jk}^i = T_{jk}^i + F_k^i \hat{V}_j - F_j^i \hat{V}_k.$$

We should be interested in obtaining a quasi-connection $\hat{\Gamma}$ with the torsion tensor field $\hat{T}$ such that

$$(26) \quad \hat{T}_{jk}^i = I_{jk}^i.$$

From (21), (25) and (26) we have

$$(27) \quad F_k^i \hat{V}_j - F_j^i \hat{V}_k = \frac{n}{n-1} [(PT)_{kj}^i - (PT)_{jk}^i].$$

Now, multiplying (27) by G_m^k , we obtain

$$(28) \quad \delta_m^i \hat{V}_j - F_j^i G_m^k \hat{V}_k = \frac{n}{n-1} [(PT)_{kj}^i - (PT)_{jk}^i] G_m^k.$$

By contracting i and m in (28) we find

$$(29) \quad \hat{V}_j = \frac{n}{(n-1)^2} [(PT)_{kj}^m - (PT)_{jk}^m] G_m^k.$$

Using (2) and (3) Eq. (29) can be written as

$$(30) \quad \hat{V}_j = -\frac{1}{n-1} T_{jk}^m G_m^k.$$

Substituting (30) in (24) we obtain the quasi-connection $\hat{\Gamma}$ given by

$$(31) \quad \hat{\Gamma}_{jk}^i = \Gamma_{jk}^i - \frac{1}{n-1} F_k^i T_{jk}^m G_m^k.$$

Let $\hat{R}_{hij}^k$ be the curvature tensor field of the quasi-connection $\hat{\Gamma}$. From Eqs. (18), (23), (30) and (31) we can conclude, by some easy computations, that the curvature tensor field of the quasi-connection $\hat{\Gamma}$ is given by

$$(32) \quad \hat{R}_{hij}^k = J_{hij}^k.$$

Hence we have proved

Theorem. *There exists a quasi-connection $\hat{\Gamma}_{jk}^i$ of the group G with the torsion tensor field $\hat{T}_{jk}^i$ and the curvature tensor field $\hat{R}_{hij}^k$ given by*

$$(33) \quad \hat{T}_{jk}^i = I_{jk}^i, \quad \hat{R}_{hij}^k = J_{hij}^k.$$

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FAMILII DE CVASI-CONEXIUNI PE VARIETĂȚI DIFERENȚIABILE (I)

(Rezumat)

Dată o varietate diferențiabilă M și un tensor nedegenerat F de tip $(1,1)$, după [2], [3] se poate introduce noțiunea de cvasi-conexiune pe M . Existența tensorului nedegenerat F permite și definirea pe M a doi operatori de tip Obata. Cu ajutorul tensorului F și a acestor doi operatori se determină pe M o familie de cvasi-conexiuni și se studiază câteva proprietăți algebrice și geometrice ale acestei familii.

GEOMETRY OF FINSLER SUBSPACES (II)

BY

AUREL BEJANCU

0. Introduction. This is the second paper in a series we would like to devote to the geometry of Finsler subspaces. In the first one we developed a new theory of Finsler subspaces by means of the concept of vectorial Finsler connection. As a final result, we obtained all the structure equations for a Finsler subspace in a Finsler space endowed with an arbitrary Finsler connection.

It is the purpose of the present paper to study the geometry of Finsler subspaces in a Finsler space endowed with the Cartan connection.

First, in § 1 we recall some fundamental results from our paper [2]. Then in § 2 we consider the Cartan connection on the ambient Finsler space and prove that all induced Finsler connections are metrical with respect to the Finsler metrics. We find in a very convenient form all the structure equations of a Finsler subspace in a Finsler space endowed with the Cartan connection. Such equations were obtained by M. M a t s u m o t o [5] and R. M i r o n [7] for Finsler subspaces of codimension one. Some results on the theory of Finsler subspaces of arbitrary codimension can be found in H. R u n d [8].

We use the notations and terminology from our paper [2] without any other notification.

1. Basic Results for Geometry of Finsler Subspaces. Let $F^m = (\tilde{M}, \tilde{L}(x, y))$ be a Finsler space, where $\tilde{M}$ is a real m -dimensional differentiable manifold and $\tilde{L}(x, y)$ is the fundamental function of F^m (see M. M a t s u m o t o [4]). Denote by $g_{ij}(x, y)$ the fundamental tensor field of $\tilde{M}$ and suppose it is positive definite.

Next, let M be an n -dimensional submanifold of $\tilde{M}$ locally given by equations

$$x^i = x^i(u^1, \dots, u^n), \quad \text{rank} \left| \frac{\partial x^i}{\partial u^\alpha} \right| = n.$$

In this section and in the sequel we use the following range of indices : $i, j, k, \dots = 1, \dots, m$; $\alpha, \beta, \gamma, \dots = 1, \dots, n$; $a, b, c, \dots = n+1, \dots, m$. The immer-

sion i of M in $\widetilde{M}$ induces an immersion di (the differential of i) of TM in $\widetilde{TM}$. Thus, if locally (u, v) of TM is carried by di in $(x(u), xy(u, v))$ of $\widetilde{TM}$ we have

$$(1.1) \quad y^i = B_\alpha^i v^\alpha, \quad \text{where} \quad B_\alpha^i = \frac{\partial x^i}{\partial u^\alpha}.$$

We have to note that (y^i) and (v^α) are the supporting elements of $\widetilde{M}$ and respectively M . We denote by L and $g_{\alpha\beta}(u, v)$ the induced fundamental function and respectively the induced fundamental tensor field of M . Then we have

$$(1.2) \quad g_{\alpha\beta}(u, v) = g_{ij}(x(u), y(u, v)) B_\alpha^i B_\beta^j.$$

Thus we are dealing with geometry of the Finsler subspace $F^n = (M, L(u, v))$ immersed in F^m . We considered in [2] the fundamental tensor field $g_{ij}(x, y)$ as a Riemannian metric on the vertical subbundle VTM of $\widetilde{TM}$. Then we defined the Finsler normal bundle to F^n as being the complementary orthogonal subbundle $VTM^\perp$ to VTM in VTM^* , where VTM and VTM^* are respectively the vertical subbundle of TTM and the restriction of VTM to TM .

Now we consider a Finsler connection $FC = (\tilde{N}, \tilde{\nabla}) = (N^j_i, F^j_k, C^j_k)$ on the ambient Finsler space F^m . We obtained in [2] three induced vectorial Finsler connections: $IVFC = (N, \tilde{\nabla}) = (N^\beta_\alpha, F^j_\alpha, C^j_\alpha)$, $IFC = (N, \nabla) = (N^\beta_\alpha, F^\beta_\gamma, C^\beta_\gamma)$ and $FNC = (N, \nabla^\perp) = (N^\beta_\alpha, F^b_\alpha, C^b_\alpha)$ on the vector bundles VTM^* , VTM and respectively $VTM^\perp$. Certainly, IFC is nothing but a Finsler connection on the Finsler space F^n . All these vectorial Finsler connections have the same non-linear connection $N = (N^\beta_\alpha)$ given by (see [1], [2])

$$(1.3) \quad N^\beta_\alpha = \tilde{B}^\beta_i (B^i_{0\alpha} + N^i_j B^j_\alpha).$$

Next, let $\left\{ B_a = B_a^i \frac{\partial}{\partial y^i} \right\}$ be a local field of orthonormal frames in the Finsler normal bundle $VTM^\perp$. Then the coefficients of the linear connections $\tilde{\nabla}$, ∇ and $\nabla^\perp$ are given by

$$(1.4) \quad \begin{cases} F^j_\alpha = F^j_k B^k_\alpha + C^j_k B^k_\alpha H^a_\alpha, & C^j_\alpha = C^j_k B^k_\alpha, \\ F^\beta_\gamma = \tilde{B}^\beta_j (B^i_{\alpha\gamma} + B^i_\alpha F^j_\gamma), & C^\beta_\gamma = C^j_\gamma \tilde{B}^\beta_j B^i_\alpha, \\ F^b_\alpha = \tilde{B}^b_i B^i_{\alpha|}, & C^b_\alpha = \tilde{B}^b_i B^i_{\alpha|}, \end{cases}$$

where „ $|$ “ and „ $|$ “ means the (h) and respectively (v) -covariant derivatives with respect to the pair $(IVFC, IFC)$ and H^a_α are given by

$$(1.5) \quad H^a_\alpha = \tilde{B}^a_i (B^i_{0\alpha} + N^i_j B^j_\alpha).$$

From 1.3) and (1.5) by using (2.10) of [2] we obtain

$$(1.6) \quad H_{\alpha}^a B_a^j + N_{\alpha}^{\beta} B_{\beta}^j = B_{0\alpha}^j + N_{\alpha}^j B_{\alpha}^i.$$

Also, we have

$$(1.7) \quad \frac{\delta}{\delta u^{\alpha}} = B_{\alpha}^i \frac{\delta}{\delta x^i} + H_{\alpha}^a B_a^{\cdot}, \quad \frac{\partial}{\partial v^{\alpha}} = B_{\alpha}^i \frac{\partial}{\partial y^i},$$

where we put

$$(1.8) \quad \frac{\delta}{\delta u^{\alpha}} = \frac{\partial}{\partial u^{\alpha}} - N_{\alpha}^{\beta} \frac{\partial}{\partial v^{\beta}}, \quad \frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N_{\alpha}^j \frac{\partial}{\partial y^j}.$$

By using the above three vectorial Finsler connections we obtained in [2] all the structure equations of the immersion of F^n in F^m .

2. The Induced Vectorial Finsler Connections by the Cartan Connection.

Throughout this section we denote by $FC^m = (F_{\alpha}^{*j_k}, N_{\alpha}^{*j_i}, C_{\alpha}^{*j_k})$ the Cartan connection on F^m , that is, we have

$$(2.1) \quad \begin{cases} F_{\alpha}^{*j_k} = \frac{1}{2} g^{jr} \left\{ \frac{\delta g_{ir}}{\delta x^k} + \frac{\delta g_{kr}}{\delta x^i} - \frac{\delta g_{ik}}{\delta x^r} \right\}, \\ C_{\alpha}^{*j_k} = \frac{1}{2} g^{jr} \frac{\partial g_{ir}}{\partial y^k}, \quad N_{\alpha}^{*j_k} = F_{\alpha}^{*j_k} y^i. \end{cases}$$

It is well-known that FC^m is both (h) and (v) -metrical and its deflection tensor $D_{\alpha}^{*j_i}$ and torsion tensors $T_{j_k}^{*i}$ and $S_{j_k}^{*i}$ are vanishing.

Now, by using (3.31) of [2] we obtain

$$(2.2) \quad g_{ij|\alpha} = g_{ij}|_{\alpha} = 0.$$

Then by differentiating covariantly (1.2) and taking into account (2.10) and (3.30) of [2] we get

$$(2.3) \quad g_{\alpha\beta|\gamma} = g_{\alpha\beta}|_{\gamma} = 0.$$

Finally, by differentiating covariantly (2.8) of [2] with respect to the pair (FNC, IFC) we obtain

$$(2.4) \quad \delta_{ab} \perp_{\alpha} = \delta_{ab} \perp_{\alpha} = 0.$$

Therefore by (2.2)–(2.4) we have

Theorem 2.1. *All the induced vectorial Finsler connections $IVFC$, IFC and FNC by the Cartan connection are both (h) -metrical and (v) -metrical in sense of [1].*

Next we consider the intrinsic Cartan connection $FC^n = (F_{\alpha}^{*\beta}_{\gamma}, N_{\alpha}^{*\beta}, C_{\alpha}^{*\beta}_{\gamma})$ on F^n and the induced Finsler connection $IFC = (F_{\alpha}^{\beta}_{\gamma}, N_{\alpha}^{\beta}, C_{\alpha}^{\beta}_{\gamma})$ on F^n by the Cartan connection FC^m on F^m . We are looking for the relationship between FC^n and IFC . In order to do this we first note that by (1.3) and (1.4) the induced Finsler connection IFC is given by

$$(2.5) \quad \begin{cases} N_{\alpha}^{\beta} = \tilde{B}_{\alpha}^{\beta} (B_{0\alpha}^j + N_{\alpha}^{*j_i} B_{\alpha}^i), \\ F_{\alpha}^{\beta}_{\gamma} = \tilde{B}_{\alpha}^{\beta} (B_{\alpha\gamma}^j + F_{\alpha}^{*j_k} B_{\alpha}^i B_{\gamma}^k + C_{\alpha}^{*j_k} B_{\alpha}^i B_{\gamma}^k H_{\gamma}^k), \\ C_{\alpha}^{\beta}_{\gamma} = \tilde{B}_{\alpha}^{\beta} C_{\alpha}^{*j_k} B_{\gamma}^k B_{\gamma}^k. \end{cases}$$

Thus we see that both the deflection tensor D_{α}^{β} and the torsion tensor $S_{\alpha\gamma}^{\beta}$ are vanishing. However, the torsion tensor $T_{\alpha\gamma}^{\beta}$ is given by

$$(2.6) \quad T_{\alpha\gamma}^{\beta} = M_{\alpha}^{\beta} H_{\gamma}^{\alpha} - M_{\gamma}^{\beta} H_{\alpha}^{\alpha},$$

where we put

$$(2.7) \quad M_{\alpha}^{\beta} = C^{\alpha}{}_{\gamma} B_{\alpha}^{\beta} \tilde{B}_{\gamma}^{\beta}.$$

Hence, generally the induced Finsler connection does not coincide with the intrinsic Cartan connection on F^n .

Lemma 2.1. *Let F^n be a Finsler subspace of the Finsler space F^m endowed with the Cartan connection. Then we have*

$$(2.8) \quad \frac{1}{2} g^{\beta\varphi} \left\{ \frac{\delta g_{\alpha\varphi}}{\delta u^{\gamma}} + \frac{\delta g_{\varphi\gamma}}{\delta u^{\alpha}} - \frac{\delta g_{\alpha\gamma}}{\delta u^{\varphi}} \right\} = F_{\alpha\gamma}^{\beta} + M_{\alpha\gamma}^{\beta} H_{\alpha}^{\alpha} - H^{a\beta} M_{a\alpha\gamma},$$

where $\frac{\delta}{\delta u^{\alpha}}$ is given by (1.7) and we put

$$(2.9) \quad H^{a\beta} = H_{\varphi}^{\alpha} g^{\varphi\beta}, \quad M_{a\alpha\gamma} = g_{\varphi\alpha} M_{a}^{\varphi\gamma}.$$

Proof. By direct computation in the left side of (2.8) by using (1.2), (1.7), (2.5) and (2.7).

We denote by $\frac{\delta^*}{\delta u^{\alpha}}$ the differential operator defined by the non-linear connection of the Cartan connection FC^n on F^n , that is, we have

$$(2.10) \quad \frac{\delta^*}{\delta u^{\alpha}} = \frac{\partial}{\partial u^{\alpha}} - N^{\beta}{}_{\alpha} \frac{\partial}{\partial v^{\beta}}.$$

Then by (1.8) and (2.10) we get

$$(2.11) \quad \frac{\delta^*}{\delta u^{\alpha}} = \frac{\delta}{\delta u^{\alpha}} + (N^{\beta}{}_{\alpha} - N^{*\beta}{}_{\alpha}) \frac{\partial}{\partial v^{\beta}}.$$

Thus by using (2.8) and (2.11) we obtain

$$(2.12) \quad F_{\alpha\gamma}^{\beta} = F_{\alpha\gamma}^{\beta} + M_{\alpha\gamma}^{\beta} H_{\alpha}^{\alpha} - H^{a\beta} M_{a\alpha\gamma} + (N^{\varepsilon}{}_{\gamma} - N^{*\varepsilon}{}_{\gamma}) C_{\alpha\varepsilon}^{*\beta} + (N^{\varepsilon}{}_{\alpha} - N^{*\varepsilon}{}_{\alpha}) C_{\gamma\varepsilon}^{*\beta} - g^{\beta\varphi} (N^{\varepsilon}{}_{\varphi} - N^{*\varepsilon}{}_{\varphi}) C_{\alpha\gamma\varepsilon}^{*}.$$

Since for both, the induced Finsler connection and the Cartan connection on F^n the deflection tensor is vanishing, from (2.12) we infer

$$(2.13) \quad N^{\beta}{}_{\gamma} = N^{\beta}{}_{\gamma} + M_{\alpha\gamma}^{\beta} H_{\alpha}^{\alpha} v^{\alpha} + (N^{\varepsilon}{}_{\alpha} - N^{*\varepsilon}{}_{\alpha}) C_{\gamma\varepsilon}^{*\beta} v^{\alpha}.$$

By contracting (2.13) with v^{γ} we get

$$(2.14) \quad (N^{\beta}{}_{\gamma} - N^{\beta}{}_{\gamma}) v^{\gamma} = 0.$$

Hence (2.13) becomes

$$(2.15) \quad N^{\beta}{}_{\gamma} = N^{\beta}{}_{\gamma} - M_{\alpha\gamma}^{\beta} H_{\alpha}^{\alpha}, \quad \text{where we put} \quad H_0^{\alpha} = H_{\alpha}^{\alpha} v^{\alpha}.$$

Next, by direct computation we get

$$(2.16) \quad g^{\beta\varphi} M_{a\varphi}^{\varepsilon} C_{\alpha\gamma\varepsilon}^* = M_{a\varepsilon}^{\beta} C_{\alpha\gamma}^{*\varepsilon}.$$

Thus by using (2.15) and (2.16) in (2.12) we obtain

$$(2.17) \quad F_{\alpha\gamma}^{\beta} = F_{\alpha\gamma}^{*\beta} + (M_{a\varepsilon}^{\beta} C_{\alpha\gamma\varepsilon}^{*\beta} + M_{a\varepsilon}^{\beta} C_{\gamma\varepsilon}^{*\beta} - M_{a\varepsilon}^{\beta} C_{\alpha\varepsilon}^{*\beta}) H_0^a - M_{a\gamma}^{\beta} H_{\alpha}^a + H^{\alpha\beta} M_{a\alpha\gamma}.$$

Finally, it is easy to check

$$(2.18) \quad C_{\alpha\gamma}^{\beta} = C_{\alpha\gamma}^{*\beta}.$$

By using (2.15), (2.17) and (2.18) which in fact show the relationship between the Finsler connections IFC and FC^n on F^n we obtain

Theorem 2.2. *Let F^n be a Finsler subspace of a Finsler space F^m endowed with Cartan connection FC^m . Then the induced Finsler connection IFC on F^n coincides with the intrinsic Cartan connection FC^n on F^n if and only if we have*

$$(2.19) \quad M_{a\alpha}^{\beta} H_{\gamma}^a = M_{a\gamma}^{\beta} H_{\alpha}^a.$$

Remark 2.1. The proof of Theorem 2.2 follows either by using the axiomatic standpoint of M. Matsuoto (see [4]) with respect to the Cartan connection. Also, formulas (2.15) and (2.17) can be obtained by Hasiuchi's method with respect to the axiomatic standpoint of Finsler connections with prescribed torsion tensor $T_{\alpha\gamma}^{\beta}$ (see [3]).

In order to get all the torsion tensors of IFC on F^n we recall that for the Cartan connection FC^m on F^m we have $P_{jk}^i = C_{jk|0}^i$ (see M. Matsuoto [4], p. 114). Then we denote

$$(2.20) \quad \begin{cases} Q_{\alpha\beta\gamma} = C_{ijk|0}^i B_{\alpha}^j B_{\beta}^k B_{\gamma}^i, & Q_{a\beta\gamma} = C_{ijk|0}^i B_{\beta}^j B_{\gamma}^k B_a^i; \\ R_{\alpha\beta\gamma} = g_{\alpha\varphi} R^{\varphi}_{\beta\gamma}, & P_{\alpha\beta\gamma} = g_{\alpha\varphi} P^{\varphi}_{\beta\gamma}, & S_{\alpha\beta\gamma} = g_{\alpha\varphi} S^{\varphi}_{\beta\gamma}. \end{cases}$$

By using (4.12) and (4.13) of [2] and (2.2) we get

$$(2.21) \quad \tilde{H}_{\alpha\beta}^a = H_{\alpha\beta}^a, \quad \tilde{K}_{\alpha\beta}^a = K_{\alpha\beta}^a.$$

Thus (4.10) and (5.15) of [2], (2.20) and (2.21) imply

$$(2.22) \quad \begin{cases} R_{\alpha\beta\gamma} = R_{ijk}^i B_{\alpha}^j B_{\beta}^k B_{\gamma}^i + H_{\beta}^a (H_{a\alpha\gamma} - Q_{a\alpha\gamma}) + H_{\gamma}^a (Q_{a\alpha\beta} - H_{a\alpha\beta}); \\ P_{\alpha\beta\gamma} = M_{a\alpha\beta} H_{\gamma}^a + Q_{\alpha\beta\gamma}, & S_{\alpha\beta\gamma} = 0. \end{cases}$$

By using the second equation in (2.22) and Theorem 2.2 we obtain

Theorem 2.3. *Let F^n be a Finsler subspace of a Finsler space F^m endowed with the Cartan connection FC^m . Then the induced Finsler connection IFC coincides with the intrinsic Cartan connection FC^n on F^n if and only if $P_{\alpha\beta\gamma}$ is a symmetric Finsler tensor field with respect to the last two indices.*

Now, in order to get a simple form of the structure equations of F^n in F^m we define the tensor fields

$$(2.23) \quad \begin{cases} R_{ij\alpha\beta} = g_{jk} R_{i\alpha\beta}^k, & P_{ij\alpha\beta} = g_{jk} P_{i\alpha\beta}^k, & S_{ij\alpha\beta} = g_{jk} S_{i\alpha\beta}^k, \\ R_{ab\alpha\beta} = \delta_{bc} R_{a\alpha\beta}^c, & P_{ab\alpha\beta} = g_{bc} P_{a\alpha\beta}^c, & S_{ab\alpha\beta} = g_{bc} S_{a\alpha\beta}^c, \\ R_{\gamma\varphi\alpha\beta} = g_{\varepsilon\varphi} R_{\gamma\alpha\beta}^{\varepsilon}, & P_{\gamma\varphi\alpha\beta} = g_{\varepsilon\varphi} P_{\gamma\alpha\beta}^{\varepsilon}, & S_{\gamma\varphi\alpha\beta} = g_{\varepsilon\varphi} S_{\gamma\alpha\beta}^{\varepsilon}, \end{cases}$$

where $(R^k_{i\alpha\beta}, P^k_{i\alpha\beta}, S^k_{i\alpha\beta})$, $(R^c_{a\alpha\beta}, P^c_{a\alpha\beta}, S^c_{a\alpha\beta})$ and $(R^e_{\gamma\alpha\beta}, P^e_{\gamma\alpha\beta}, S^e_{\gamma\alpha\beta})$ are the curvature tensors of vectorial Finsler connections *IVFC*, *FNC*, and respectively *IFC* (see [2]). Moreover, by (5.3) of [2] we get

$$(2.24) \quad \begin{cases} R_{ij\alpha\beta} = R^*_{ijkh} B^k_{\alpha} B^h_{\beta} + P^*_{ijkh} (B^k_{\alpha} H^a_{\beta} - B^k_{\beta} H^a_{\alpha}) B^h_{\alpha} + S^*_{ijkh} B^k_{\alpha} B^h_{\beta} H^a_{\alpha} H^h_{\beta}, \\ P_{ij\alpha\beta} = P^*_{ijkh} B^k_{\alpha} B^h_{\beta} + S^*_{ijkh} B^k_{\alpha} H^a_{\beta} B^h_{\alpha}, \\ S_{ij\alpha\beta} = S^*_{ijkh} B^k_{\alpha} B^h_{\beta}, \end{cases}$$

where $(R^*_{ijkh}, P^*_{ijkh}, S^*_{ijkh})$ are the curvature tensors of the Cartan connection FC^m on F^m . By using (17.9) of [4], (2.23) and (2.24) we obtain

$$(2.25) \quad R_{ij\alpha\beta} = -R_{ji\alpha\beta}, \quad P_{ij\alpha\beta} = -P_{ji\alpha\beta}, \quad S_{ij\alpha\beta} = -S_{ji\alpha\beta}.$$

Thus by (2.21) and (2.25) we can easily check that in fact the equations (5.7) and (5.9) coincide for the particular case of the Cartan connection on F^m . Therefore, by (5.6)–(5.9) of [2] we have

Theorem 2.4. *Let F^n be a Finsler subspace of the Finsler space F^m endowed with Cartan connection FC^m . Then the structure equations of the immersion of F^n in F^m are the following*

$$(2.26) \quad \begin{cases} R_{ij\alpha\beta} B^i_{\gamma} B^j_{\varphi} = R_{\gamma\varphi\alpha\beta} + H^a_{\gamma\beta} H_{a\varphi\alpha} - H^a_{\gamma\alpha} H_{a\varphi\beta}, \\ P_{ij\alpha\beta} B^i_{\gamma} B^j_{\varphi} = P_{\gamma\varphi\alpha\beta} + K^a_{\gamma\beta} H_{a\varphi\alpha} - H^a_{\gamma\alpha} K_{a\varphi\beta}, \\ S_{ij\alpha\beta} B^i_{\gamma} B^j_{\varphi} = S_{\gamma\varphi\alpha\beta} + K^a_{\gamma\beta} K_{a\varphi\alpha} - K^a_{\gamma\alpha} K_{a\varphi\beta}; \end{cases}$$

$$(2.27) \quad \begin{cases} R_{ij\alpha\beta} B^i_{\gamma} B^j_{\alpha} = H_{a\gamma\alpha\beta} - H_{a\gamma\beta\alpha} + H_{a\gamma\varphi} T^{\varphi}_{\alpha\beta} + K_{a\gamma\varphi} R^{\varphi}_{\alpha\beta}, \\ P_{ij\alpha\beta} B^i_{\gamma} B^j_{\alpha} = H_{a\gamma\alpha\beta} - K_{a\gamma\beta\alpha} + H_{a\gamma\varphi} C^{\varphi}_{\alpha\beta} + K_{a\gamma\varphi} P^{\varphi}_{\alpha\beta}, \\ S_{ij\alpha\beta} B^i_{\gamma} B^j_{\alpha} = K_{a\gamma\alpha\beta} - K_{a\gamma\beta\alpha}; \end{cases}$$

$$(2.28) \quad \begin{cases} R_{ij\alpha\beta} B^i_{\alpha} B^j_{\beta} = R_{ab\alpha\beta} + H_{b\varphi\alpha} H^{\varphi}_{\beta} - H_{b\varphi\beta} H^{\varphi}_{\alpha}, \\ P_{ij\alpha\beta} B^i_{\alpha} B^j_{\beta} = P_{ab\alpha\beta} + H_{b\varphi\alpha} K^{\varphi}_{\beta} - K_{a\varphi\beta} H^{\varphi}_{\alpha}, \\ S_{ij\alpha\beta} B^i_{\alpha} B^j_{\beta} = S_{ab\alpha\beta} + K_{b\varphi\alpha} K^{\varphi}_{\beta} - K_{b\varphi\beta} K^{\varphi}_{\alpha}. \end{cases}$$

The equations (2.26), (2.27) and (2.28) are called the *equations of Gauss*, *Codazzi* and respectively *Ricci* for the immersion of F^n in F^m when we consider on the ambient Finsler space F^m the Cartan connection. By using (3.25), (3.27) and (4.10) of [2] we obtain the second fundamental forms of the immersion of F^n in F^m given by

$$(2.29) \quad \begin{cases} H_{a\alpha\beta} = B^j_{\alpha} (g_{ij} B^i_{\beta} + F^*_{ijk} B^k_{\alpha} B^j_{\beta}) + M_{ab\alpha} H^b_{\beta}, \\ K_{a\alpha\beta} = M_{a\alpha\beta}, \end{cases}$$

where we put $M_{ab\alpha} = C^*_{ijk} B^i_{\alpha} B^j_{\beta} B^k_{\alpha}$.

Remark 2.2. Some authors were concerned with relationships between curvature tensor fields of the Cartan connections on F^n and F^m . These could be obtained from (2.26) by using (2.15), (2.17), (2.18) and (2.24).

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GEOMETRIA SUBSPAȚIILOR FINSLER (II)

(Rezumat)

În [1] și [2] autorul a obținut o teorie generală a subspațiilor Finsler într-un spațiu Finsler dotat cu o conexiune Finsler arbitrară. În prezenta lucrare se dezvoltă această teorie pentru cazul în care conexiunea spațiului Finsler ambiant este conexiunea Cartan. Se obțin toate ecuațiile de structură ale subspațiului și se studiază relația dintre conexiunea indusă și conexiunea Cartan intrinsecă a subspațiului.

THEOREMS OF DECOMPOSITION FOR ALMOST SEMI-INVARIANT SUBMANIFOLDS IN A SASAKIAN MANIFOLD

BY

N. PAPAGHIUC

0. Introduction. Almost semi-invariant submanifolds have been introduced by A. B e j a n c u and the present author in [1]. For the geometry of an almost semi-invariant submanifold it is of great interest the study of the distribution $\tilde{D}$ (see § 1). For this reason we are concerned in this paper with some problems on this distribution. More precisely, by means of a condition on the morphism b (see Definition 2.1) we obtain theorems of decomposition for an almost semi-invariant submanifold in a Sasakian manifold (Theorems 2.1 and 2.2) or in a Sasakian space form (Theorem 2.3).

1. Definitions and Basic Formulas. Let $\tilde{M}$ be a $2n+1$ -dimensional almost contact metric manifold with (F, ξ, η, g) as the almost contact metric structure, where F is a tensor field of type (1.1), ξ is a vector field, η is a 1-form and g is a Riemannian metric of $\tilde{M}$. These tensor fields satisfy

$$(1.1) \quad F^2 = -I + \eta \otimes \xi, \quad F(\xi) = 0, \quad \eta \circ F = 0, \quad \eta(\xi) = 1;$$

$$(1.2) \quad g(FX, FY) = g(X, Y) - \eta(X)\eta(Y)$$

for all vector fields X, Y tangent to $\tilde{M}$, where I denotes the identity morphism on the tangent bundle $T\tilde{M}$.

All the manifolds and morphisms in this paper are considered to be differentiable of class C^∞ .

The Levi-Civita connection on $\tilde{M}$ is denoted by $\tilde{\nabla}$. Then, it is known that $\tilde{M}$ is a Sasakian manifold if and only if we have ([4], p. 73)

$$(1.3) \quad (\tilde{\nabla}_X F)Y = g(X, Y)\xi - \eta(Y)X$$

for all X, Y tangent to $\tilde{M}$. Also, from (1.3) follows

$$(1.4) \quad \tilde{\nabla}_X \xi = -FX, \quad \text{for any } X \text{ tangent to } \tilde{M}.$$

Let M be an m -dimensional Riemannian manifold isometrically immersed in a Sasakian manifold $\tilde{M}$. Denote by TM and $TM^\perp$ the tangent

bundle of M and respectively the normal bundle to M . Suppose the structure vector field ξ of $\widetilde{M}$ be tangent to the submanifold M and denote by $\{\xi\}$ the 1-dimensional distribution spanned by ξ on M and by $\{\xi\}^\perp$ the complementary orthogonal distribution to $\{\xi\}$ in TM .

For any vector bundle H on M we denote by $\Gamma(H)$ the module of all differentiable sections of H .

For all $X \in \Gamma(TM)$ we have $g(FX, \xi) = 0$. Then we put

$$(1.5) \quad FX = bX + cX,$$

where $bX \in \Gamma(\{\xi\}^\perp)$ and $cX \in \Gamma(TM^\perp)$. Thus b is an endomorphism of the tangent bundle TM and c is a normal bundle valued 1-form on M .

Next, for each $x \in M$ we define the following subspaces:

$$(1.6) \quad D_x = \{X_x \in \{\xi_x\}^\perp; \quad c(X_x) = 0\},$$

$$(1.7) \quad D_x^\perp = \{X_x \in \{\xi_x\}^\perp; \quad b(X_x) = 0\}.$$

We note that D_x and $D_x^\perp$ are two orthogonal subspaces of the tangent space $T_x M$.

Definition 1.1. The submanifold M of the Sasakian manifold $\widetilde{M}$ is said to be an almost semi-invariant submanifold if $\dim D_x$ and $\dim D_x^\perp$ are constant along M and $D: x \rightarrow D_x \subset T_x M$; $D^\perp: x \rightarrow D_x^\perp \subset T_x M$ define differentiable distributions on M .

Then we have

Proposition 1.1. [1] a. The distribution D is the maximal invariant distribution in $\{\xi\}^\perp$, that is, we have

i) $FD_x = D_x$ for any $x \in M$ and

ii) if $D'_x \subset \{\xi_x\}^\perp$ and $FD'_x = D'_x$ then we have $D'_x \subset D_x$ for any $x \in M$.

b. The distribution $D^\perp$ is the maximal anti-invariant distribution in $\{\xi\}^\perp$, that is, we have

i) $F(D_x^\perp) \subset T_x M^\perp$ for any $x \in M$ and

ii) if $D''_x \subset \{\xi_x\}^\perp$ and $F(D''_x) \subset T_x M^\perp$ then we have $D''_x \subset D_x^\perp$ for any $x \in M$.

Now we denote by $\tilde{D}$ the complementary orthogonal distribution to $D \oplus D^\perp \oplus \{\xi\}$ in TM . Thus for the tangent bundle to the almost semi-invariant submanifold M we have the orthogonal decomposition

$$(1.8) \quad TM = D \oplus D^\perp \oplus \tilde{D} \oplus \{\xi\}.$$

We note that $\tilde{D}$ is neither an invariant nor an anti-invariant distribution on M and $\dim \tilde{D}_x = 2s$ for any $x \in M$ [2].

We denote $\dim D_x = 2p$ and $\dim D_x^\perp = q$. Then, there exists the following particular cases of almost semi-invariant submanifolds:

1. semi-invariant submanifolds when $s=0$, $p \neq 0$, $q \neq 0$ [3];
2. invariant submanifolds when $q=s=0$, $p \neq 0$ [5];
3. anti-invariant submanifolds when $p=s=0$, $q \neq 0$ [6];
4. almost invariant submanifolds when $q=0$, $p \neq 0$, $s \neq 0$ [1];

5. *almost anti-invariant submanifolds* when $p=0, q \neq 0, s \neq 0$ [1];

6. *pseudo-invariant submanifolds* when $p=q=0; s \neq 0$ [1].

Fundamental results on the geometry of an almost semi-invariant submanifold in a Sasakian manifold are obtained in [1].

The purpose of the present paper is to obtain theorems of decomposition for an almost semi-invariant submanifold in a Sasakian manifold. In order to do this, we recall some formulas and results from [1].

Let M be an almost semi-invariant submanifold in the Sasakian manifold $\tilde{M}$. We denote by g both Riemannian metrics on $\tilde{M}$ and M . Also, the projection morphisms of TM to the distributions $D, D^\perp$, and $\tilde{D}$ are denoted respectively by P, Q and L . Then, for each $X \in \Gamma(TM)$ we have

$$(1.9) \quad X = PX + QX + LX + \eta(X)\xi, \quad \text{where} \quad \eta(X) = g(X, \xi).$$

We take $X \in \Gamma(\tilde{D})$ and note that $bX \neq 0$ and $cX \neq 0$. Thus c defines a vector subbundled $c\tilde{D} : x \rightarrow c\tilde{D}_x$ of $TM^\perp$ and $\dim c\tilde{D}_x = 2s$.

For any $N \in \Gamma(TM^\perp)$ we put

$$(1.10) \quad FN = tN + fN,$$

where tN is the tangential part of FN and fN is the normal part of FN . We note that we have

$$(1.11) \quad g(FD^\perp, c\tilde{D}) = 0.$$

Next, we denote by γ the orthogonal complementary vector bundle to $FD^\perp \oplus c\tilde{D}$ in $TM^\perp$. Thus we have

$$(1.12) \quad TM^\perp = FD^\perp \oplus c\tilde{D} \oplus \gamma.$$

Lemma 1.1 [1]. *The morphisms t and f satisfy*

$$(1.13) \quad t(TM^\perp) = D^\perp \oplus \tilde{D}, \quad t(FD^\perp) = D^\perp, \quad t(c\tilde{D}) = \tilde{D} \quad \text{and} \quad f(c\tilde{D}) = c\tilde{D}.$$

Definition 1.2. The almost semi-invariant submanifold M of the Sasakian manifold $\tilde{M}$ is said to be generic if we have $\gamma = \{0\}$.

Lemma 1.2 [1]. *Let M be an almost semi-invariant submanifold of the Sasakian manifold $\tilde{M}$. Then the endomorphism $b : TM \rightarrow TM$ is an f -structure on TM , that is $b^3 + b = 0$ (see K. Yano [7]) if and only if M is a semi-invariant submanifold. If $s \neq 0$, then we have $(b^3 + b + tcb)X = 0$ for any $X \in \Gamma(TM)$.*

Now, we denote by ∇ the Riemannian connection on M with respect to the metric g . The linear connection induced by $\tilde{\nabla}$ on the normal bundle $TM^\perp$ is denoted by $\nabla^\perp$. Then the equations of Gauss and Weingarten are given by

$$(1.14) \quad \tilde{\nabla}_X Y = \nabla_X Y + h(X, Y)$$

and respectively

$$(1.15) \quad \tilde{\nabla}_X N = -A_N X + \nabla_X^\perp N$$

for all $X, Y \in \Gamma(TM)$ and $N \in \Gamma(TM^\perp)$, where h is the second fundamental form of M and A_N is the fundamental tensor of Weingarten with respect to the normal section N . These tensor fields are related by

$$(1.16) \quad g(h(X, Y), N) = g(A_N X, Y)$$

for all $X, Y \in \Gamma(TM)$ and $N \in \Gamma(TM^\perp)$.

Definition 1.3. The almost semi-invariant submanifold M of the Sasakian manifold $\widetilde{M}$ is called

- i) D -geodesic if and only if we have $h(X, Y) = 0$ for any $X, Y \in \Gamma(D)$.
- ii) $(D, \widetilde{D})$ -geodesic if and only if we have $h(X, Z) = 0$ for all $X \in \Gamma(D)$ and $Z \in \Gamma(\widetilde{D})$.

For all $X, Y \in \Gamma(TM)$ we denote

$$(1.17) \quad u(X, Y) = \nabla_X F P Y + \nabla_X b L Y - A_{\tau_Q Y} X - A_{c_L Y} X.$$

Then we have

Lemma 1.3 [1]. *Let M be an almost semi-invariant submanifold of the Sasakian manifold $\widetilde{M}$. Then we have*

$$(1.18) \quad P(u(X, Y)) = F P(\nabla_X Y) - \eta(Y) P X;$$

$$(1.19) \quad Q(u(X, Y)) = Q(h(X, Y)) - \eta(Y) Q X;$$

$$(1.20) \quad L(u(X, Y)) = b L(\nabla_X Y) + L t(h(X, Y)) - \eta(Y) L X;$$

$$(1.21) \quad \eta(u(X, Y)) = g(F X, F Y);$$

$$(1.22) \quad \begin{aligned} h(X, F P Y) + h(X, b L Y) + \nabla_X^{\frac{1}{2}} F Q Y + \nabla_X^{\frac{1}{2}} c L Y = \\ = F Q \nabla_X Y + c L \nabla_X Y + f(h(X, Y)) \end{aligned}$$

for all $X, Y \in \Gamma(TNM)$;

$$(1.23) \quad \nabla_X \xi = -F X, \quad h(X, \xi) = 0 \quad \text{for any } X \in \Gamma(D);$$

$$(1.24) \quad \nabla_Y \xi = 0, \quad h(Y, \xi) = -F Y \quad \text{for any } Y \in \Gamma(D^\perp);$$

$$(1.25) \quad \nabla_Z \xi = -b Z, \quad h(Z, \xi) = -c Z \quad \text{for any } Z \in \Gamma(\widetilde{D});$$

$$(1.26) \quad \nabla_\xi \xi = 0, \quad h(\xi, \xi) = 0;$$

$$(1.27) \quad \nabla_\xi U \in \Gamma(D) \quad \text{for any } U \in \Gamma(D);$$

$$(1.28) \quad \nabla_\xi V \in \Gamma(D^\perp) \quad \text{for any } V \in \Gamma(D^\perp);$$

$$(1.29) \quad \nabla_\xi W \in \Gamma(\widetilde{D}) \quad \text{for any } W \in \Gamma(\widetilde{D}).$$

2. Theorems of Decomposition. Let M be an almost semi-invariant submanifold of the Sasakian manifold $\widetilde{M}$. The purpose of this paragraph is to study the fundamental properties of the endomorphism b and to obtain theorems of decomposition for the submanifold M .

First, by direct computation using (1.2), (1.3), (1.9) and (1.18)–(1.21) we obtain

Lemma 2.1. *Let M be an almost semi-invariant submanifold of a Sasakian manifold $\widetilde{M}$. Then we have*

$$(2.1) \quad th(X, Y) = (\nabla_X b)Y - A_{c_Y}X - (\tilde{\nabla}_X F)Y$$

for all $X, Y \in \Gamma(TM)$, where $(\nabla_X b)Y = \nabla_X bY - b\nabla_X Y$.

The endomorphism b is said to be parallel if we have $\nabla_X b = 0$ for any $X \in \Gamma(TM)$.

Proposition 2.1. *Let M be an almost semi-invariant submanifold of a Sasakian manifold $\widetilde{M}$. If the endomorphism b is parallel then M is an anti-invariant submanifold of $\widetilde{M}$.*

Proof. We take $Y \in \Gamma(D)$ in (2.1) and by using (1.3) obtain $th + g(X, Y)\xi = 0$ which implies

$$(2.2) \quad g(X, Y) = 0 \quad \text{for all } X \in \Gamma(TM) \quad \text{and } Y \in \Gamma(D).$$

From (2.2) follows that M is an almost anti-invariant submanifold. On the other hand, from $\nabla_X b = 0$ we have

$$(2.3) \quad \eta(\nabla_X bY) = 0 \quad \text{for all } X, Y \in \Gamma(TM).$$

By using (1.25) in (2.3) we obtain

$$(2.4) \quad g(bX, bY) = 0 \quad \text{for all } X, Y \in \Gamma(\tilde{D}).$$

Finally, the proof follows from (2.2) and (2.4).

It is interesting to find weaker conditions than that one of b being parallel and obtain theorems of decomposition for proper almost semi-invariant submanifolds.

Definition 2.1. Let M be an almost semi-invariant submanifold of a Sasakian manifold $\widetilde{M}$. The endomorphism $b : TM \rightarrow TM$ is said to be η -parallel if for all $X, Y \in \Gamma(TM)$ we have

$$(2.5) \quad (\nabla_X b)Y = g(PX, PY)\xi + g(bLX, bLY)\xi - \eta(Y)PX + \eta(Y)b^2LX.$$

Lemma 2.2. *Let M be an almost semi-invariant submanifold of a Sasakian manifold $\widetilde{M}$. Then, the endomorphism b is η -parallel if and only if we have*

$$(2.6) \quad th(X, Y) = \eta(Y)QX - \eta(Y)tcLX - QA_{c_Y}X - LA_{c_Y}X$$

for all $X, Y \in \Gamma(TM)$.

Proof. First, by using (1.2) and (1.5) we have

$$(2.7) \quad g(FLX, cY) = g(cLX, cLY) = g(LX, LY) - g(bLX, bLY) \quad \text{and}$$

$$(2.8) \quad g(FQX, cY) = g(FQX, FQY) = g(QX, QY) \quad \text{for all } X, Y \in \Gamma(TM).$$

Then, by using (1.3), (1.9), (1.16), (2.7), (2.8) and the fact that $(b^2 + tc)LX = -LX$ in (2.1) we obtain

$$(2.9) \quad th(X, Y) = \{(\nabla_X b)Y - g(PX, PY)\xi - g(bLX, bLY)\xi + \eta(Y)PX - \eta(Y)b^2LX\} + \{\eta(Y)QX - \eta(Y)tcLX - QA_{c_Y}X - LA_{c_Y}X - PA_{c_Y}X\}.$$

On the other hand, from (2.6), by using (1.16) we have

$$(2.10) \quad PA_{cY}X=0 \quad \text{for all} \quad X, Y \in \Gamma(TM).$$

Then, the proof follows from (2.9) and (2.10).

Now, we state

Theorem 2.1. *Let M be a $2p+q+2s+1$ -dimensional generic almost semi-invariant submanifold of a $2p+2q+4s+1$ -dimensional Sasakian manifold $\tilde{M}$. If the endomorphism b is η -parallel then M is locally the Riemannian direct product $M_1 \times M_2$ where M_1 is a $2p+2s+1$ -dimensional almost invariant submanifold of $\tilde{M}$ totally geodesic immersed in M and D -geodesic and $(D, \tilde{D})$ -geodesic immersed in $\tilde{M}$ and M_2 is a q -dimensional anti-invariant submanifold of $\tilde{M}$ such that the structure vector field ξ is normal to M_2 .*

Proof. Since M is a generic semi-invariant submanifold, we have $\gamma=\{0\}$, therefore $t: TM^\perp \rightarrow D^\perp \oplus \tilde{D}$ is an isomorphism. Then, from (2.6) follows

$$(2.11) \quad h(X, Y)=0 \quad \text{for all} \quad X \in \Gamma(TM) \quad \text{and} \quad Y \in \Gamma(D).$$

For any $Y \in \Gamma(D^\perp)$, by using (2.5) we obtain

$$(2.12) \quad b\nabla_X Y=0 \quad \text{for all} \quad X \in \Gamma(TM),$$

which implies $\nabla_X Y \in \Gamma(D^\perp \oplus \{\xi\})$ for any $X \in \Gamma(TM)$ and $Y \in \Gamma(D^\perp)$.

On the other hand, from (1.23)–(1.25) we have

$$(2.13) \quad \eta(\nabla_X Y)=g(\nabla_X Y, \xi)=-g(Y, \nabla_X \xi)=g(Y, bX)=0$$

for all $Y \in \Gamma(D^\perp)$ and $X \in \Gamma(TM)$. From (2.12) and (2.13) follows $\nabla_X Y \in \Gamma(D^\perp)$ for all $X \in \Gamma(TM)$ and $Y \in \Gamma(D^\perp)$ which means the distribution $D^\perp$ is parallel. Now, by using (2.11) in (1.22) we obtain

$$(2.14) \quad FQ\nabla_X Y=h(X, bLY)+\nabla_X^{\frac{1}{2}}cLY-cL\nabla_X Y-f(h(X, Y))$$

for any $X \in \Gamma(TM)$ and $Y \in \Gamma(D \oplus \tilde{D} \oplus \{\xi\})$.

Taking account that the distribution $D^\perp$ is parallel, we take $N=FZ$, $Z \in \Gamma(D^\perp)$ and obtain

$$(2.15) \quad g(FQ\nabla_X Y, FZ)=g(\nabla_X Y, Z)=-g(Y, \nabla_X Z)=0$$

for all $X \in \Gamma(TM)$, $Y \in \Gamma(D \oplus \tilde{D} \oplus \{\xi\})$ and $Z \in \Gamma(D^\perp)$. From (2.15) follows $FQ\nabla_X Y=0$ for all $X \in \Gamma(TM)$ and $Y \in \Gamma(D \oplus \tilde{D} \oplus \{\xi\})$, that is $\nabla_X Y \in \Gamma(D \oplus \tilde{D} \oplus \{\xi\})$ for all $X \in \Gamma(TM)$ and $Y \in \Gamma(D \oplus \tilde{D} \oplus \{\xi\})$ which means the distribution $D \oplus \tilde{D} \oplus \{\xi\}$ is parallel too.

Therefore M is locally the Riemannian direct product $M_1 \times M_2$ where M_1 is a leaf of the distribution $D \oplus \tilde{D} \oplus \{\xi\}$ and M_2 is a leaf of $D^\perp$. Of course M_1 is an almost invariant submanifold of M and M_2 is an anti-invariant submanifold such that ξ is normal to M_2 . Moreover, from (2.14) we obtain

$$(2.16) \quad h(X, bY)+\nabla_X^{\frac{1}{2}}cY=cL\nabla_X Y+fh(X, Y) \in \Gamma(c\tilde{D})$$

for all $X \in \Gamma(TM)$ and $Y \in \Gamma(D \oplus \tilde{D} \oplus \{\xi\})$.

Thus, the condition (4.9) of Theorem 4.2 of [1] is satisfied, hence M_1 is totally geodesic immersed in M . Also, from (2.11) and (2.16) follows that M_1 is D -geodesic and $(D, \tilde{D})$ -geodesic immersed in $\tilde{M}$. The proof is complete.

Theorem 2.2. *Let M be a $2p+q+2s+1$ -dimensional almost semi-invariant submanifold of a $2p+2q+4s+2k+1$ -dimensional Sasakian manifold $\tilde{M}$, where $\dim \gamma_x = 2k$ for any $x \in M$. If the endomorphism b is η -parallel then M is locally the Riemannian direct product $M_1 \times M_2$ where M_1 is a $2p+2s+1$ -dimensional almost invariant submanifold of $\tilde{M}$ totally geodesic immersed in M and M_2 is a q -dimensional anti-invariant submanifold such that the structure vector field ξ is normal to M_1 .*

Proof. In a similar way as in Theorem 2.1 we obtain

$$(2.17) \quad h(X, bY) + \nabla_X^{\perp} cY = cLV_X Y + fh(X, Y) - h(X, FPY) \in \Gamma(c\tilde{D} \oplus \gamma)$$

for all $X \in \Gamma(TM)$ and $Y \in \Gamma(D \oplus \tilde{D} \oplus \{\xi\})$. Then, the assertion of Theorem 2.2 follows from Theorem 4.4 and Theorem 4.2 of [1].

Now, suppose $\tilde{M}(k)$ be a Sasakian space form of constant F -sectional curvature k and M be an almost semi-invariant submanifold of $\tilde{M}(k)$. The curvature tensor $\tilde{R}$ of $\tilde{M}(k)$ is given by

$$(2.18) \quad \begin{aligned} \tilde{R}(X, Y)Z = & \frac{k+3}{4} \{g(Y, Z)X - g(X, Z)Y\} + \frac{k-1}{4} \{\eta(X)\eta(Z)Y - \\ & - \eta(Y)\eta(Z)X + g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi + \\ & + g(Z, FY)FX - g(Z, FX)FY + 2g(X, FY)FZ\} \end{aligned}$$

for all X, Y, Z tangent to $\tilde{M}(k)$. Then, the equation of Codazzi is given by

$$(2.19) \quad \begin{aligned} (D_X h)(Y, Z) - (D_Y h)(X, Z) = & \frac{k-1}{4} \{g(Z, bY)cX - \\ & - g(Z, bX)cY + 2g(X, bY)cZ\} \end{aligned}$$

for all X, Y, Z tangent to M , where $(D_X h)(Y, Z)$ is defined by

$$(2.20) \quad (D_X h)(Y, Z) = \nabla_X^{\perp}(h(Y, Z)) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z).$$

Theorem 2.3. *Let M be a $2p+q+2s+1$ -dimensional generic almost semi-invariant submanifold of a $2p+2q+4s+1$ -dimensional simply connected and complete Sasakian space form $\tilde{M}(k)$. If the endomorphism b is η -parallel then we have either :*

1. M is an almost anti-invariant submanifold of $\tilde{M}(k)$ and M is $M_1 \times M_2$ where M_1 is a $2s+1$ -dimensional pseudo-invariant submanifold of $\tilde{M}(k)$ and M_2 is a q -dimensional anti-invariant submanifold of $\tilde{M}(k)$ or

2. $k = -3$ and M is $M_1 \times M_2$ of $R^{2p+q+4s+1}(-3)$ where M_1 is a $2p+2s+1$ -dimensional almost invariant submanifold totally geodesic immersed in M and D -geodesic and $(D, \tilde{D})$ -geodesic immersed in $R^{2p+2q+4s+1}(-3)$ and M_2 is a q -dimensional anti-invariant submanifold of $R^{2p+2q+4s+1}(-3)$ and ξ is normal to M_2 .

Proof. By using (2.11) and (2.20) we obtain

$$(D_X h)(Y, Z) = -h(Y, \nabla_X Z) \text{ for all } X, Y \in \Gamma(TM) \text{ and } Z \in \Gamma(D).$$

Then, from (2.19) follows

$$(2.21) \quad \frac{k-1}{4} \{g(Z, bY)cX - g(Z, bX)cY\} = h(X, \nabla_Y Z) - h(Y, \nabla_X Z)$$

for all $X, Y \in \Gamma(TM)$ and $Z \in \Gamma(D)$. In (2.21) we take $Y \in \Gamma(D)$, $Z = FY$ and obtain

$$(2.22) \quad \frac{k-1}{4} g(FY, FY)cX = h(X, \nabla_Y FY) \text{ for all } X \in \Gamma(TM).$$

Since the endomorphism b is η -parallel we have the distribution $D \oplus \tilde{D} \oplus \{\xi\}$ is parallel. Then, from (1.22) and (2.11) we obtain $L\nabla_X Z = 0$ for all $X, Z \in \Gamma(D)$ and (2.22) becomes

$$(2.23) \quad \frac{k-1}{4} g(FY, FY)cX = -\eta(\nabla_Y FY)cX$$

for any $X \in \Gamma(TM)$ and $Y \in \Gamma(D)$.

By using (1.23) in (2.23) we obtain

$$(2.24) \quad \frac{k+3}{4} g(FY, FY)cX = 0$$

for all $X \in \Gamma(TM)$ and $Y \in \Gamma(D)$.

Taking account that M is a generic almost semi-invariant submanifold, from (2.24) we have $k = -3$ or $FY = 0$ for any $Y \in \Gamma(D)$. If $FY = 0$ for any $Y \in \Gamma(D)$, then M is a $2s+q+1$ -dimensional almost anti-invariant submanifold of $\tilde{M}(k)$ and the assertion 1 follows from Theorem 2.1. If $k = -3$, then the ambient manifold $\tilde{M}(k)$ is $R^{2p+2q+4s+1}(-3)$ and the assertion 2 follows also from Theorem 2.1.

Remark. By $R^{2n+1}(-3)$ we mean the Sasakian space form with constant F -sectional curvature $k = -3$ with standard Sasakian structure in the Euclidean space R^{2n+1} (cf. [4], p. 99).

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TEOREME DE DESCOMPUNERE PENTRU SUBVARIETĂȚI APROAPE SEMIINVARIANTE ÎNTR-O VARIETATE SASAKI

(Rezumat)

Se continuă lucrarea [1] obținind condiții suficiente ca o subvarietate aproape semi-invariantă a unei varietăți Sasaki să se descompună local ca produsul Riemannian al unei subvarietăți aproape invariantă cu o subvarietate anti-invariantă (teoremele 2.1 și 2.2). Pentru cazul particular al subvarietăților aproape semi-invariante a unei forme spațiale Sasaki se demonstrează teorema 3.2.

LIMITE ET CONTINUITÉ ANALYTIQUES DES MULTI-APPLICATIONS (III)

PAR

AL. SABADIȘ

1. **Propriétés des multi-applications analytiquement continues.** Dans cet travail on va noter par $\mathcal{O}(E, Y, x_0)$ la famille des multi-applications (E, Y, F) , (a) -continues dans le point $x_0 \in E \subset X$, X étant un espace de convergence.

Proposition 1. *L'ensemble $\mathcal{O}(E, Y, x_0)$ muni avec les opérations de réunion, intersection et complémentarité, est une sousalgèbre Boole de $L(E, Y, x_0)$.*

Démonstration. On voit immédiatement que $\mathcal{O}(E, Y, x_0) \subset L(E, Y, x_0)$. On démontre maintenant que les trois opérations sont internes. Pour tout $F, G \in \mathcal{O}(E, Y, x_0)$ on a

$$1) \quad (a)\text{-}\lim_{x \rightarrow x_0} (F \cup G)(x) = (a)\text{-}\lim_{x \rightarrow x_0} F(x) \cup G(x) = [(a)\text{-}\lim_{x \rightarrow x_0} F(x)] \cup$$

$$\cup [(a)\text{-}\lim_{x \rightarrow x_0} G(x)] = F(x_0) \cup G(x_0) = (F \cup G)(x_0) \Rightarrow F \cup G \in \mathcal{O}(E, Y, x_0).$$

$$2) \quad (a)\text{-}\lim_{x \rightarrow x_0} (F \cap G)(x) = (a)\text{-}\lim_{x \rightarrow x_0} F(x) \cap G(x) = [(a)\text{-}\lim_{x \rightarrow x_0} F(x)] \cap$$

$$\cap [(a)\text{-}\lim_{x \rightarrow x_0} G(x)] = F(x_0) \cap G(x_0) = (F \cap G)(x_0) \Rightarrow F \cap G \in \mathcal{O}(E, Y, x_0).$$

$$3) \quad (a)\text{-}\lim_{x \rightarrow x_0} \bar{F}(x) = (a)\text{-}\lim_{x \rightarrow x_0} [Y - F(x)] = Y - (a)\text{-}\lim_{x \rightarrow x_0} F(x) =$$

$$= Y - F(x_0) = \bar{F}(x_0) \Rightarrow \bar{F} \in \mathcal{O}(E, Y, x_0).$$

Proposition 2. *Si $F, G \in \mathcal{O}(E, Y, x_0)$ alors $F - G \in \mathcal{O}(E, Y, x_0)$ et $F \Delta G \in \mathcal{O}(E, Y, x_0)$.*

Démonstration. On a

$$(a)\text{-}\lim_{x \rightarrow x_0} (F - G)(x) = (a)\text{-}\lim_{x \rightarrow x_0} [F(x) - G(x)] = [(a)\text{-}\lim_{x \rightarrow x_0} F(x)] - [(a)\text{-}\lim_{x \rightarrow x_0} G(x)] =$$

$$= F(x_0) - G(x_0) = (F - G)(x_0) \Rightarrow F - G \in \mathcal{O}(E, Y, x_0).$$

Du $F \Delta G = (F - G) \cup (G - F)$ il résulte que $F \Delta G \in \mathcal{O}(E, Y, x_0)$ et la proposition est démontrée.

Proposition 3. Si la multi-application (E, Y, F) est analytiquement continue dans le point $x_0 \in E$ et si existent $A, B \in P(Y)$ tels que $A \subset F(x_0) \subset B$, alors il existe $n_0 \in N$ tel que $A \subset F(x_n) \subset B$ pour tout suite $(x_n) \subset E$, $x_n \rightarrow x_0$ et $n > n_0$.

Démonstration. Par hypothèse on a $(a)\text{-}\lim_{x \rightarrow x_0} F(x) = F(x_0)$. Donc pour tout suite $(x_n) \subset E$, $x_n \rightarrow x_0$, on a

$$F(x_n) \Delta F(x_0) = [F(x_n) - F(x_0)] \cup [F(x_0) - F(x_n)] \subseteq D_n \downarrow \Phi, \quad n > n_0.$$

D'ici on a la relation

$$F(x_0) - D_n \subseteq F(x_n) \subseteq F(x_0) \subseteq D_n, \quad \text{pour } n > n_0.$$

De la suite (D_n) on peut éliminer chaque point après un nombre fini de pas. C'est pourquoi on peut bien conclure qu'il existe $n_0 \in N$ tel que $n > n_0 \Rightarrow A \subset F(x_0) - D_n \subseteq F(x_n) \subseteq F(x_0) \cup D_n \subset B$, ce qui démontre la proposition.

Proposition 4. Si les multi-applications (E, Y, F) et (E, Y, G) sont analytiquement continues dans le point $x_0 \in E$ et $F(x_0) \subset G(x_0)$, alors pour tout suite $(x_n) \subset E$, $x_n \rightarrow x_0$, il existe $n_0 \in N$ tel que $n > n_0 \Rightarrow F(x_n) \subset G(x_n)$.

Démonstration. On peut prendre $A \in P(Y)$ tel que $F(x_0) \subset A \subset G(x_0)$. Vu la Proposition 3 on aura $n_1, n_2 \in N$ tels que $n > n_1 \Rightarrow F(x_n) \subset A$ et $n > n_2 \Rightarrow A \subset G(x_n)$, $\forall (x_n) \subset E$, $x_n \rightarrow x_0$. Donc, pour $n > \max\{n_1, n_2\}$ on aura $F(x_n) \subset A \subset G(x_n)$ ce qui démontre l'affirmation.

Proposition 5. Soient X un espace topologique et Y un ensemble quelconque. Si la multi-application (E, Y, F) , $E \subset X$, a les propriétés :

a) pour tout $B \in P(Y)$ il existe $x \in E$ tel que $F(x) = B$;

b) $A \subset E$ est un ensemble séquentiellement compact ;

c) F est analytiquement continue sur A , alors $F(A) \subset Y$ est séquentiellement compact.

Démonstration. Pour toute suite $(A_n) \subset F(A)$, il existe une suite $(x_n) \subset A$ telle que $F(x_n) = A_n$. La suite $(x_n) \subset A$ contienne, vu l'hypothèse b), une suite partielle (x_{n_k}) convergente à $x_0 \in A$. Pour la suite (x_{n_k}) , $x_{n_k} \rightarrow x_0 \in A$, on aura $F(x_{n_k}) \xrightarrow{(a)} F(x_0) \subset F(A)$, vu l'hypothèse c). Du $(F(x_{n_k})) \subset (F(x_n)) = (A_n) \subset F(A)$, il résulte que $(F(x_{n_k}))$ est une suite partielle de la suite (A_n) , analytiquement convergente à $B = F(x_0) \subset F(A)$. Donc $F(A)$ est un ensemble séquentiellement compact.

2. Notion de multi-application uniformément continue dans le sens analytique. On considère la multi-application (E, Y, F) , où $E \subset X$, X étant un espace de convergence et Y un ensemble quelconque. La multi-application F s'appelle analytiquement continue sur $A \subset E$ si elle est analytiquement continue dans chaque point de l'ensemble A .

Donc, on peut écrire

$$[F \text{ est analytiquement continue sur } A \subset E] \Leftrightarrow [\forall x_0 \in A, (a)\text{-}\lim_{x \rightarrow x_0} F(x) = F(x_0)] \Leftrightarrow [\forall x_0 \in A \text{ et } \forall (x_n) \subset A, x_n \rightarrow x_0 \Rightarrow F(x_n) \xrightarrow{(a)} F(x_0)] \Leftrightarrow [\forall x_0 \in A \text{ et } \forall (x_n) \subset A, x_n \rightarrow x_0 \Rightarrow \exists (D_n^{x_0}) \subset P(Y), D_n^{x_0} \downarrow \Phi, \text{ telle que } F(x_n) \Delta F(x_0) \subseteq D_n^{x_0}].$$

Définition 1. La multi-application (E, Y, F) est uniformément continue dans le sens analytique sur $A \subset E$, si pour toute suite $(x_n) \subset A$

$x_n \rightarrow x_0$, il existe une suite d'ensembles $(D_n) \subset P(Y)$, $D_n \downarrow \Phi$, indépendante de x_0 , telle que $F(x_n)\Delta F(x_0) \subseteq D_n$.

Observation. Si l'ensemble A est fini, $A = \{x_1, x_2, \dots, x_k\}$, alors une multi-application (E, Y, F) analytiquement continue sur A est uniformément continue dans le sens analytique sur A . En effet, pour $x_0 = x_k$ il existe $(D_n^{x_k}) \subset (P(Y))$, $D_n^{x_k} \downarrow \Phi$, telle que $F(x_n)\Delta F(x_k) \subset D_n^{x_k}$, pour tout suite $(x_n) \subset A$, $x_n \rightarrow x_k$.

On sait qu'une réunion finie de suites d'ensembles descendant à l'ensemble vide est, aussi, descendante à l'ensemble vide. Puisque pour tout suite $(x_n) \subset A$, $x_n \rightarrow x_p$, $p = \overline{1, k}$, on aura $F(x_n)\Delta F(x_p) \subseteq \bigcup_{p=1}^k D_n^{x_p} = D_n \downarrow \Phi$; il résulte que F est uniformément continue dans le sens analytique sur A .

Exemple 1. La multi application (E, Y, F) définie par $F(x) = \bar{B}$, quelque soit $x \in E$, est uniformément continue sur E parce que pour tout $(x_n) \subset E$, $x_n \rightarrow x \in E$, on a $F(x_n)\Delta F(x) = A\Delta A = \Phi \downarrow \Phi$, quelque soit $x \in E$.

Exemple 2. On considère l'application (X, X, F) définie par $F(x) = \{x\}$, pour tout $x \in X$. Pour une suite $(x_n) \subset X$, $x_n \rightarrow x \in X$, on a $F(x_n)\Delta F(x) = \{x_n\}\Delta \{x\} = \{x_n\} \cup \{x\} = \{x_n, x\}$. La suite $\{x_1, x\}$, $\{x_2, x\}$, ..., $\{x_n, x\}$, n'est pas descendante à l'ensemble vide, parce que on ne peut pas éliminer l'élément x après un nombre fini de pas. Donc, F est analytiquement discontinue dans chaque point de X . Vu la Définition 1 on ne pose pas le problème de la continuité uniforme dans le sens analytique sur X .

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LIMITA ȘI CONTINUITATEA ANALITICĂ A MULTIFUNCȚIILOR (III)

(Rezumat)

Se demonstrează unele proprietăți ale multifuncțiilor analitic continue într-un punct și se introduce conceptul de multifuncție uniform continuă în sens analitic pe o mulțime.

INEQUALITIES INVOLVING IMPROPER INTEGRALS

BY

ADRIAN CORDUNEANU

Recently, V.M. Singare [1] (cf. Math. Rev., 85, f. 26 020) considered the integral inequality

$$(1) \quad u(x, y) \leq f(x, y) + a(x, y) F \left(\int_x^\infty \int_y^\infty b(s, t) g(u(s, t)) ds dt \right), \quad x, y \geq 0$$

and obtained several bounds for the function $u = u(x, y)$, under different restrictions that those imposed in the present article on the involved functions u, f, a, F, b and g . Inequalities in two independent variables of the form

$$(2) \quad u(x, y) \leq f(x, y) + a(x, y) \int_x^\infty \int_y^\infty b(s, t) g(u(s, t)) ds dt, \quad x, y \geq 0,$$

were considered by other authors. The case $f(x, y) = \text{const.}$, $a(x, y) = 1$ and $g(u) = u$ was firstly investigated by T. Nurimov (cf. [2], pp. 76-77), who obtained for u a majoration of Wendroff type. In the case of the particular inequality

$$(3) \quad u(x, y) \leq c + \int_x^\infty \int_y^\infty b(s, t) g(u(s, t)) ds dt, \quad x, y \geq 0,$$

the conditions imposed on the function g are weaker than in the general case. We will denote by $G = G(u)$ a function satisfying the relation $G'(u) = 1/g(u)$ for $u > 0$. We remark that G is strictly increasing for $u > 0$. We also denote by $G(\infty)$ the limit (finite or infinite) of this function when $u \rightarrow \infty$. Of course, it is supposed that the improper integral appearing in the right hand side of (3) is convergent and moreover

$$(4) \quad \int_0^\infty \int_0^\infty b(s, t) ds dt < \infty.$$

Proposition 1. Assume that in (3) $u=u(x, y)$ and $b=b(x, y)$ are continuous nonnegative functions for $x, y \geq 0$ and that $c > 0$ is a constant. Assume also that $g=g(u)$ is continuous for $u \geq 0$, positive for $u > 0$, $g'(u)$ is continuous and nonnegative for $u > 0$ and (4) is satisfied. Then we have

$$(5) \quad u(x, y) \leq G^{-1} \left(G(c) + \iint_{x, y}^{\infty, \infty} b(s, t) ds dt \right),$$

provided that

$$(6) \quad G(c) + \iint_{x, y}^{\infty, \infty} b(s, t) ds dt < G(\infty).$$

Proof. Denoting

$$(7) \quad h(x, y) = c + \iint_{x, y}^{\infty, \infty} b(s, t) g(u(s, t)) ds dt, \quad H(x, y) = G(h(x, y))$$

we obtain after a simple computation

$$(8) \quad H''_{st}(s, t) \leq b(s, t), \quad s, t \geq 0.$$

Integrating (8) on the domain $x \leq s < \infty, y \leq t < \infty$ it follows

$$(9) \quad G(h(x, y)) \leq G(c) + \iint_{x, y}^{\infty, \infty} b(s, t) ds dt$$

and the conclusion holds. We point out that if $G(\infty) = \infty$, the majoration (5) is true for all $x, y \geq 0$.

Remark. 1. If u, b and c are as in the Proposition 1, then from the inequality

$$(10) \quad u(x, y) \leq c + \iint_{x, y}^{\infty, \infty} b(s, t) u(s, t) ds dt, \quad x, y \geq 0$$

we deduce the following bound for the function u

$$(11) \quad u(x, y) \leq c \exp \left(\iint_{x, y}^{\infty, \infty} b(s, t) ds dt \right), \quad x, y \geq 0.$$

The next proposition deals with the inequalities of the form

$$(12) \quad u(x, y) \leq f(x, y) + a(x, y) \iint_{x, y}^{\infty, \infty} b(s, t) g(u(s, t)) ds dt, \quad x, y \geq 0.$$

Proposition 2. Suppose that $u(x, y)$, $b(x, y)$ and $g(u)$ are as in the Proposition 1 and that $g(u)/\lambda \leq g(u/\lambda)$ for $\lambda \geq 1$, $u \geq 0$. Suppose also that $f(x, y) \geq 1$ is continuous nonincreasing for $x, y \geq 0$ and that $a(x, y)$ is continuous nonnegative for $x, y \geq 0$, having continuous partial derivatives $a'_x \leq 0$, $a'_y \leq 0$, $a''_{xy} \geq 0$. Then from (12) it follows

$$(13) \quad u(x, y) \leq f(x, y) G^{-1} \left(G(1) + a(x, y) \iint_{x, y}^{\infty, \infty} b(s, t) ds dt \right),$$

provided that (6) is satisfied.

Proof. Denoting $u(x, y)/f(x, y) = v(x, y)$, we can write

$$(14) \quad v(x, y) \leq 1 + a(x, y) \iint_{x, y}^{\infty, \infty} b(s, t) g(v(s, t)) ds dt, \quad x, y \geq 0,$$

where the improper integral of the right hand side is convergent. If we put

$$(15) \quad h(x, y) = 1 + a(x, y) \iint_{x, y}^{\infty, \infty} b(s, t) g(v(s, t)) ds dt$$

and consider the function $H(x, y) = G(h(x, y))$, we obtain after a simple computation that

$$(16) \quad H''_{st}(s, t) \leq A''_{st}(s, t), \quad s, t \geq 0,$$

where $A(x, y)$ is given by

$$(17) \quad A(x, y) = a(x, y) \iint_{x, y}^{\infty, \infty} b(s, t) ds dt, \quad x, y \geq 0.$$

Integrating (16) on the domain $x \leq s < \infty$, $y \leq t < \infty$, taking into account (15) and the fact that $A(x, y) \rightarrow 0$ when $x \rightarrow \infty$ or $y \rightarrow \infty$, we conclude that

$$(18) \quad G(h(x, y)) \leq G(1) + a(x, y) \iint_{x, y}^{\infty, \infty} b(s, t) ds dt$$

and consequently the bound (13) is true, provided that (6) holds. Of course, if $G(\infty) = \infty$ the majoration (13) is true for all $x, y \geq 0$.

Remark 2. An example of function $g = g(u)$ which appears in the preceding Proposition 2 is $g(u) = u^\alpha$, with $0 < \alpha \leq 1$. Examples of functions $a = a(x, y)$ are $a = a_1(x) + a_2(y)$, $a = a_1(x)a_2(y)$ where a_i , $i = 1, 2$ have continuous ≤ 0 derivatives of the first order.

Proposition 3. Assume that in (12) $u(x, y)$, $b(x, y)$ and $g(u)$ are as in the Proposition 1, g is submultiplicative and $g(u)/\lambda \leq g(u/\lambda)$ for $0 < \lambda \leq 1$, $u \geq 0$. Assume also that the continuous nonincreasing function $f = f(x, y)$ sa-

satisfies one of the conditions $0 < \alpha \leq f(x, y) \leq 1$ or $0 \leq f(x, y) \leq \alpha < 1$. Then the bound (13) remains true, provided that (6) is satisfied.

The proof of the above statement is similar to those given for the Proposition 2. Suppose that $0 < \alpha \leq f(x, y) \leq 1$. We use the condition that g is submultiplicative to insure that the improper integral in (14) is convergent. Indeed, we have

$$(19) \quad g(v(s, t)) = g(u(s, t)/f(s, t)) \leq g(u(s, t)/\alpha) \leq g(1/\alpha)g(u(s, t))$$

and we remark that in fact we need only a condition weaker than submultiplicativity, namely

$$(20) \quad g(\lambda u) \leq g(\lambda)g(u) \quad \text{for } u \geq 0, \lambda \geq 1.$$

In the case $0 \leq f(x, y) \leq \alpha < 1$, we choose $\varepsilon > 0$ such that $\alpha + \varepsilon < 1$ and define the function $v(x, y)$ as $u(x, y)/(f(x, y) + \varepsilon)$. Because

$$(21) \quad g(v(s, t)) \leq g(u(s, t)/\varepsilon) \leq g(1/\varepsilon)g(u(s, t))$$

we again conclude that the integral in (14) is finite. Thus we obtain

$$(22) \quad u(x, y) \leq (f(x, y) + \varepsilon)G^{-1}\left(G(1) + a(x, y) \int_x^\infty \int_y^\infty b(s, t) ds dt\right)$$

provided that (6) is fulfilled and the desired conclusion holds making $\varepsilon \rightarrow 0$ (for fixed x, y) in (22).

Remark 3. An example of function $g(u)$ having all the properties required in the Proposition 3 is $g(u) = u^\beta$ with $\beta \geq 1$.

Finally, let us consider the inequality (1) and suppose that

(i) $u(x, y)$, $b(x, y)$ are continuous nonnegative for $x, y \geq 0$; $f(x, y) \geq 1$, $a(x, y) \geq 1$ are continuous and f is nonincreasing;

(ii) $g = g(u)$ is submultiplicative, continuous for $u \geq 0$, positive for $u > 0$, $g'(u)$ is continuous and nonnegative for $u > 0$ and $g(u)/\lambda \leq g(u/\lambda)$ for $\lambda \geq 1$, $u \geq 0$;

(iii) $F = F(r)$ is continuous nonnegative for $r \geq 0$ and nondecreasing, having continuous derivative $F'(r)$ for $r > 0$;

$$(iv) \quad \int_0^\infty \int_0^\infty b(x, y)g(a(x, y))dx dy < \infty.$$

Denote by $H = H(r)$ the function given by

$$(23) \quad H(r) = \int_0^r \frac{dv}{g(1+F(v))}, \quad r \geq 0$$

and observe that H is strictly increasing for $r \geq 0$, having continuous derivatives of the first and second order.

Proposition 4. Suppose that in (1) all the conditions (i)–(iv) are satisfied. Then it follows

$$(24) \quad u(x, y) \leq f(x, y) a(x, y) \left[1 + F \left(H^{-1} \left(\iint_{x, y}^{\infty, \infty} b(s, t) g(a(s, t)) ds dt \right) \right) \right]$$

provided that

$$(25) \quad \iint_{x, y}^{\infty, \infty} b(s, t) g(a(s, t)) ds dt < H(\infty).$$

Proof. Denoting $u(x, y)/f(x, y) = v(x, y)$, we have from (1)

$$(26) \quad v(x, y) \leq a(x, y)(1 + F(r)), \quad x, y \geq 0,$$

where

$$(27) \quad r(x, y) = \iint_{x, y}^{\infty, \infty} b(s, t) g(v(s, t)) ds dt, \quad x, y \geq 0.$$

We also have

$$(28) \quad r''_{xy}(x, y) = b(x, y) g(v(x, y)) \leq b(x, y) g(a(x, y)) g(1 + F(r))$$

from which we obtain

$$(29) \quad \frac{\partial}{\partial y} \left(\frac{\partial r / \partial x}{g(1 + F(r))} \right) \leq b(x, y) g(a(x, y)), \quad x, y \geq 0.$$

Integrating this inequality on the interval $[y, \infty)$ we get

$$(30) \quad - \frac{\partial r / \partial x}{g(1 + F(r))} \leq \int_y^{\infty} b(x, t) g(a(x, t)) dt$$

and a new integration gives

$$(31) \quad H(r(x, y)) \leq \iint_{x, y}^{\infty, \infty} b(s, t) g(a(s, t)) ds dt.$$

Now, the desired conclusion holds from (26) and (31). This method of proof was suggested by the work [3] of C.C. Yeh. We remark that the Propositions 2 and 4 are independent, because in the Proposition 2 it is not assumed that $a(x, y) \geq 1$.

Corollary 4.1. Assume that u, a, b are continuous nonnegative functions defined for $x, y \geq 0$, f is continuous nonincreasing and $f(x, y) \geq \alpha > 0$, $a(x, y) \geq 1$ and

$$(32) \quad \iint_{0, 0}^{\infty, \infty} a(x, y) b(x, y) dx dy < \infty.$$

Then from

$$(33) \quad u(x, y) \leq f(x, y) + a(x, y) \iint_{x, y}^{\infty, \infty} b(s, t) u(s, t) ds dt, \quad x, y \geq 0$$

it follows

$$(34) \quad u(x, y) \leq f(x, y) a(x, y) \exp \left(\iint_{x, y}^{\infty, \infty} a(s, t) b(s, t) ds dt \right), \quad x, y \geq 0.$$

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INEGALITĂȚI CONȚINÎND INTEGRALE IMPROPRII

(Rezumat)

Se stabilesc majorări de tip Bellman-Bihari pentru funcții de două variabile independente care satisfac anumite inegalități integrale, în care apar și integrale improprii. Metoda se poate extinde și la cazul funcțiilor cu mai multe variabile independente decît două.

BRIOT-BOUQUET EQUATIONS WITH PARTIAL FRÉCHET DERIVATIVES IN BANACH SPACES

BY

I. ONCIULESCU and DRAGOȘ ȘTEFANCU

1. Introduction. In our paper [4] we have studied the existence and uniqueness problems of an analytical equation in a Banach space. On the other hand, in [1] generalized Briot-Bouquet type equations have been investigated.

Our aim is now to consider such equations —i.e. Briot-Bouquet type— in Banach spaces by an analogous method as in [4].

2. Briot-Bouquet Equations with Partial Fréchet Derivatives in Banach Spaces. Let be :

a) X a linear normed space and Y a Banach space over the complex field $\mathbb{C}$;

b) $h(x)$ a linear continuous functional on X ;

c) $x_{0i}, x_{0jk} \in X, i = \overline{1, n}, j, k = \overline{1, n}$, such that $h(x_{0i}) = 1, x \in X \times X \times \dots \times X$ (n -times), $h(x) = h(x_1) \dots h(x_n)$; $\alpha_{ij} \in \mathbb{C}, i, j = \overline{1, n}$; $b_i \in Y, i = \overline{1, n}$;

d) $P(x, w, u) = \sum_{|v|+|j|+|t| \geq 2} (h(x))^v p_{vjt}(w, u)$, an analytical function in a neighbourhood of the origin of $X \times Y \times Y^{n^*}$, where $v = (v_1, \dots, v_n), t = (t_1, \dots, t_n)$ are multiindices, $|v| = v_1 + \dots + v_n, |t| = t_1 + \dots + t_n, (v_i, t_i \text{ integers } \geq 0)$;

$(h(x))^v = (h(x_1))^{v_1} \dots (h(x_n))^{v_n}$; $w \in Y, u = (u_1, \dots, u_n) \in Y^{n^*}, p_{vjt} : Y \times Y^{n^*} \rightarrow Y$ are homogeneous continuous polynomials j -th degree in w, t_1 -th degree in $u_1, \dots, t_n$ -th degree in u_n .

The equation which we take in consideration will be

$$\begin{aligned} (1) \quad & \sum_{i=1, n} (\alpha_{i1} h(x_1) + \dots + \alpha_{in} h(x_n)) \frac{\partial w}{\partial x_i}(x, x_{0i}) = \\ & = \beta_0 w + \sum_{i=1, n} b_i h(x_i) + P\left(x, w, h(x_k) \frac{\partial w}{\partial x_j}(x, x_{0jk})\right). \end{aligned}$$

For this Eq. (1) we shall study the existence and uniqueness of analytical solutions having the form

$$(2) \quad w(x) = h(x_1)c_1 + \dots + h(x_n)c_n + \sum_{|v| \geq 2} (h(x))^v v_{c_v}.$$

First we assume that the matrix $A=(\alpha_{ij})$, (α_{ij} from (1)), has a normal Jordan form with μ submatrices of orders $r_1, \dots, r_\mu$ respectively, so that $\alpha_{ii}=\lambda_{j(i)}$,

$$j(i)=\begin{cases} 1 & 1 \leq i \leq r_1, \\ p & r_1+1 \leq i \leq r_1+\dots+r_p; \\ \alpha_{j-1,j}=\varepsilon_j=\begin{cases} 1 & 2 \leq j, \quad j \neq r_1+1, r_1+r_2+1, \dots, r_1+\dots+r_{\mu-1}+1, \\ 0 & j=r_1+1, r_1+r_2+1, \dots, r_1+\dots+r_{\mu-1}+1 \end{cases} \end{cases}$$

and $\alpha_{ij}=0$ if $j > i$ or $i < j-1$.

We denote also $\varepsilon_0=0$. Then, Eq. (1) can be written in the form

$$(1') \quad \lambda_1 h(x_1) \frac{\partial w}{\partial x_1}(x, x_{01}) + \sum_{i=2, n} (\varepsilon_i h(x_{i-1}) + \lambda_{j(i)} h(x_i)) \frac{\partial w}{\partial x_i}(x, x_{0i}) =$$

$$= \beta_0 w + \sum_{i=1, n} b_i h(x_i) + P\left(x, w, h(x_k) \frac{\partial w}{\partial x_j}(x, x_{0jk})\right).$$

Substituting (2) in (1'), taking into account that $h(x_{0i})=1$, we obtain

$$(3) \quad \begin{aligned} & \lambda_1 h(x_1) c_1 + \sum_{i=2, n} (\varepsilon_i h(x_{i-1}) + \lambda_{j(i)} h(x_i)) c_i + \\ & + \sum_{|v| \geq 2} \sum_{i=2, n} \varepsilon_i v_i (h(x))^{v+\mathbf{n}_{i-1}} c_v + \sum_{|v| \geq 2} \left(\sum_{i=1, n} v_i \lambda_{j(i)} \right) h(x)^v c_v = \\ & = \beta_0 \left(\sum_{i=1, n} h(x_i) c_i + \sum_{|v| \geq 2} (h(x))^v c_v \right) + \sum_{i=1, n} b_i h(x_i) + \\ & + P\left(x, w, h(x_k) \frac{\partial w}{\partial x_j}(x, x_{0jk})\right), \end{aligned}$$

where $\mathbf{n}_1=(1, 0, \dots, 0)$, $\mathbf{n}_2=(0, 1, 0, \dots, 0)$, $\dots$, $\mathbf{n}_n=(0, \dots, 0, 1)$ (see also [1]).

2.1. *Determination of the coefficients c_k .* From (3) we obtain the following system for c_k , $k=1, n$:

$$(4) \quad \begin{aligned} & (\lambda_1 - \beta_0) c_1 = b_1 - c_2, \quad (\lambda_1 - \beta_0) c_2 = b_2 - c_3, \dots, \\ & (\lambda_1 - \beta_0) c_{r_1-1} = b_{r_1} - c_{r_1}, \quad (\lambda_1 - \beta_0) c_{r_1} = b_{r_1}, \\ & (\lambda_2 - \beta_0) c_{r_1+1} = b_{r_1+1} - c_{r_1+2}, \dots, \\ & (\lambda_2 - \beta_0) c_{r_1+r_2-1} = b_{r_1+r_2-2} - c_{r_1+r_2}, \\ & (\lambda_2 - \beta_0) c_{r_1+r_2} = b_{r_1+r_2}, \dots, \\ & (\lambda_\mu - \beta_0) c_{r_1+\dots+r_{\mu-1}+1} = b_{r_1+\dots+r_{\mu-1}+1} - c_{r_1+\dots+r_{\mu-1}+2}, \dots, \\ & (\lambda_\mu - \beta_0) c_{r_1+\dots+r_{\mu-1}} = b_{r_1+\dots+r_{\mu-1}} - c_{r_1+\dots+r_{\mu-1}}, \\ & (\lambda_\mu - \beta_0) c_{r_1+\dots+r_\mu} = b_{r_1+\dots+r_\mu}. \end{aligned}$$

If

$$(H_1) \quad \lambda_i - \beta_0 = 0, \quad i = \overline{1, n},$$

then, from (4), we can deduce $c_k, k = \overline{1, n}$.

If

$$(H_1) \quad \lambda_j - \beta_0 \quad \text{for some } j \in \{1, 2, \dots, \mu\} \text{ and } b_{r_1 + \dots + r_j} = 0,$$

then c_{r_j} can be chosen arbitrary.

2.2. Determination of the coefficients c_v with $|v| \geq 2$. Consider a multi-index $(n_1, \dots, n_\mu)$ in which n_i are integers ≥ 0 and $n_1 + \dots + n_\mu \geq 2$. Consider also the set

$$V_{n_1 \dots n_\mu} = \{v = (v_1, \dots, v_n), v_i \text{ integers } \geq 0, v_1 + \dots + v_r = n_1, \\ v_{r_1+1} + \dots + v_{r_1+r_2} = n_2, \dots, v_{r_1+r_2+\dots+r_{\mu-1}+1} + \dots + v_n = n_\mu\}.$$

In the set $V_{n_1 \dots n_\mu}$ we consider the following ordering: we shall define $v > u$ if $v_1 > u_1$ or there exists $k, 1 \leq k < n$, so that $v_i = u_i, i = \overline{1, k}$, and $v_{k+1} > u_{k+1}$, ($v = (v_1, \dots, v_n), u = (u_1, u_2, \dots, u_n)$).

Now, if L is the cardinal of the set $V_{n_1 \dots n_\mu}$, we write

$$V_{n_1 \dots n_\mu} = \{(v, 1), \dots, (v, L)\} \quad \text{where} \quad (v, 1) > (v, 2) > \dots > (v, L).$$

Then, for $c_{(v, 1)}, \dots, c_{(v, L)}$ we obtain from (3) the system

$$\begin{aligned} B_0(n)c_{(v, 1)} + D_{(2, 1)}c_{(v, 2)} + \dots + D_{(L, 1)}c_{(v, L)} &= f_1(c_p, p), \\ B_0(n)c_{(v, 2)} + \dots + D_{(L, 2)}c_{(v, L)} &= f_2(c_p, p), \\ &\vdots \\ B_0(n)c_{(v, L)} &= f_L(c_p, p), \end{aligned} \quad (5)$$

where $B_0(n) = n_1 \lambda_1 + \dots + n_\mu \lambda_\mu - \beta_0$, $D_{(i, j)}$ are integers ≥ 0 and $f_i(c_p, p)$ are sums of polynomials $(h(x_{0j_1}))^{t_1} \dots (h(x_{0j_n}))^{t_n} p_{ij_t}$ (in polar form) from the expression of P , in some c_p with $|\rho| < n_1 + \dots + n_\mu$. If

$$(H_2) \quad B_0(n) \neq 0 \quad \text{for all systems } (n_1, \dots, n_\mu)$$

with $n_1 + \dots + n_\mu = N$, $N = 2, 3, \dots$, then from (5) one obtains uniquely the coefficients c_v , with $|v| \geq 2$.

3. The Convergence of the Formal Solution (2). The convergence of the formal solution (2), obtained by the systems (3), (5), can be proved by the majoration method used in [1]. As a consequence of (H_2) , we can find a positive constant $\sigma > 2$ such that

$$(6) \quad B_0(n) \geq \sigma |v| \quad \text{for every } v = (v_1, \dots, v_n) \quad \text{with } |v| \geq 2.$$

If we denote $\tau_v = \sum_{j=1, n} (\sigma - \varepsilon_j) v_j$, then $B_0(n) \geq \sigma |v| > \tau_v \geq |v| \geq v_j$.

Consider $C_1, \dots, C_n$ such that $C_j \geq \|c_j\|$,

$$j = \overline{1, n}, \text{ and } P^*(z, y, u) = \sum_{|v| + |j| + |t| \geq 2} \|p_{vjt}\| z^v y^j u^t \text{ in which}$$

$$z^v = z_1^{v_1} \dots z_n^{v_n}, u^t = u_1^{t_1} \dots u_n^{t_n}, z_i, u_j \in \mathbb{C}.$$

Continuing we determine the formal series

$$(7) \quad y(z_1, \dots, z_n) = \sum_{j=1, n} C_j z_j + \sum_{|v| \geq 2} C_v z^v,$$

such that

$$\sum_{|v| \geq 2} \sigma |v| C_v z^v - \sum_{|v| \geq 2} \sum_{j=1, n} \varepsilon_j v_j C_v z^{v+n_j-1-n_j} =$$

$$= P^* \left(z, \sum_{j \geq 1, n} C_j z_j + \sum_{|v| \geq 2} \tau_v C_v z^v, C_k z_k + \sum_{|v| \geq 2} \tau_v C_v z^{v+n_j-n_k} \right) |h(x_{0jk})|.$$

For C_v with $v \in V_{n_1 \dots n_{\mu}}$, we obtain the system

$$\sigma |v| C_{(v, 1)} - D_{(2, 1)} C_{(v, 2)} - \dots - D_{(L, 1)} C_{(v, L)} = \bar{f}_1(\tau_\rho C_\rho, p^*),$$

$$\vdots$$

$$\sigma |v| C_{(v, L-1)} - D_{(L, L-1)} C_{(v, L)} = \bar{f}_{L-1}(\tau_\rho C_\rho, p^*),$$

$$\sigma |v| C_{(v, L)} = \bar{f}_L(\tau_\rho C_\rho, p^*),$$

where $\bar{f}_i(\tau_\rho C_\rho, p^*)$ are polynomials in $\tau_\rho C_\rho$, $|\rho| < n_1 + \dots + n_\mu$, with coefficients from the set $\{ \|p_{ijt}\| |h(x_{0i1})|^{l_1} \dots |h(x_{0jn})|^{l_n} \}$, p_{ijt} being the polynomials of the expressions $f_i(c_\rho, p)$ from (5).

From (5) and (8) it results that $\|C_{(v, k)}\| \leq C_{(v, k)}$.

As in [1] one can prove that the series (7) with the coefficients of (8) is convergent. Thus we have proved the following

Theorem. *In the hypotheses (H_1) , (H_2) the equation (1') has a unique analytical solution (2). In the hypotheses (H'_1) , (H_2) , the equation (1') has an infinity of analytical solutions (2).*

Remarks 1. If the matrix $A = (a_{ij})$, a_{ij} from (1), is not initially in Jordan form, then, by a suitable transformation $x_i = \beta_{i1} x'_1 + \dots + \beta_{in} x'_n$, $i = \overline{1, n}$, $\beta_{ij} \in \mathbb{C}$, we can arrive to the form (1') and the result is the same as that stated in the above theorem.

2. If the conditions (H_1) , (H_2) or (H'_1) , (H_2) are satisfied, then for the equations of the form

$$(1'') \quad \lambda_1 h_1(x_1) \frac{\partial w}{\partial x_1}(x, x_{01}) + \sum_{i=2, n} (\varepsilon_i h_i(x_{i-1}) + \lambda_{j(i)} h_i(x_i)) \frac{\partial w}{\partial x_i}(x, x_{0i}) =$$

$$= \beta_0 w + \sum_{i=1, n} b_i h_i(x_i) + P \left(x, w, h_k(x_k) \frac{\partial w}{\partial x_j}(x, x_{0jk}) \right),$$

where $h_i(x_i)$, $i = \overline{1, n}$, are linear continuous functionals on X and $h_i(x_{0i}) = 1$, one can determine analytical solutions

$$(2') \quad w(x) = \sum_{i=1, n} h_i(x_i) c_i + \sum_{|v| \geq 2} (h(x))^v c_v,$$

$((h(x))^v = (h_1(x_1))^{v_1} \dots (h_n(x_n))^{v_n})$, by the same method.

3. If the conditions (H_1) , (H_2) or (H'_1) , (H_2) are satisfied, then for the equation

$$(1''') \quad \lambda_1 h(x_1) \frac{\partial w}{\partial x}(x, z) + \sum_{i=2, n} (\varepsilon_i h(x_{i-1}) + \lambda_{j(i)} h(x_i)) \frac{\partial w}{\partial x_i}(x, z) =$$

$$= \left[\beta_0 w + \sum_{i=1, n} b_i h(x_i) + P \left(x, w, h_k(x_k) \frac{\partial w}{\partial x_j}(x, x_{0jk}) \right) \right] h(z),$$

where $z \in X$ is variable, an analogous method lead to analytical solution (2).

4. To analogous results one arrives if in (1'') β_0 is a linear continuous operator, $\beta_0 : Y \rightarrow Y$, so that

$$(H_3) \quad \lambda_i I - \beta_0, \quad i = \overline{1, n}, \quad I \text{ the identical operator,}$$

are invertibles and

$$(H_4) \quad |\lambda_1 n_1 + \dots + \lambda_\mu n_\mu| \geq \|\beta_0\| \quad \text{for every} \quad (n_1, \dots, n_\mu),$$

with n_i intergers ≥ 0 and $n_1 + \dots + n_\mu \geq 2$.

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ECUAȚII BRIOT-BOUQUET CU DERIVATE PARȚIALE FRÉCHET ÎN SPAȚII BANACH

(Rezumat)

Pentru ecuații de forma (1), care generalizează în spații Banach ecuațiile studiate în [1] se demonstrează existența și unicitatea unor soluții analitice de forma (2).

where λ is variable, an analogous method lead to analytical solution (2).
 4. To analogous results one arrives if in (1) λ is a linear continuous operator, $\lambda^2 = Y$, so that

$$(H) \quad \lambda^2 - \lambda a_1 = I, \quad I \text{ the identical operator,}$$

are invertibles and

$$(H') \quad \lambda^2 a_1 + \dots + \lambda^2 a_{n-1} = I, \quad \text{for every } \lambda \in \sigma(\lambda).$$

with n integers, $n \geq 0$ and $a_1 + \dots + a_n \neq 0$.

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NOTES ON THE CONTRIBUTORS TO THE SPECIAL ISSUE

(continued)

For the Special Issue, the authors (1) have been selected in special French editions in 1967, as demonstrated, especially in the following, as follows (2).

LES TYPES DE CIRCULATIONS DANS LES GRAPHS ÉVENTAILS (III)

PAR

VIOREL GHIUȘ

Soit un graphe éventail [3] (fig. 1) où $P_i, i=0, 1, \dots, p, R_j, j=1, 2, \dots, p$ sont des arcs non orientés ou des chemins simples; les flèches du graphe indiquent le sens de la circulation qu'on va tout de suite définir.

On se propose d'étudier la circulation dans ce graphe non orienté, en sachant que :

$$(1) \quad \begin{cases} 0 \leq m_i \leq F(P_i) \leq M_i, \\ 0 \leq l_j \leq F(R_j) \leq L_j, \\ i=0, 1, \dots, p, \\ j=1, 2, \dots, p. \end{cases}$$

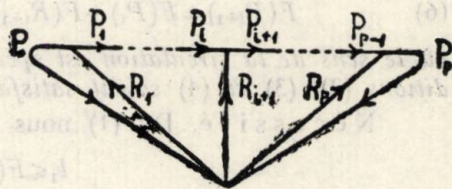


Fig. 1

Si $m_0 + m_1 < l_1$ on définit

$$\begin{cases} m'_0 = a \geq m_0, \\ m'_1 = b \geq m_1, \end{cases} \quad \text{avec } a+b=l_1 \quad \text{et} \\ \begin{cases} m''_0 = c, \\ m''_1 = d, \end{cases} \quad \text{avec } \begin{cases} a \leq c \leq M_0, \\ b \leq d \leq M_1, \end{cases} \quad \text{et } c+d \leq L_1.$$

Donc $m_0 + m_1 < l_1 = a+b \leq c+d \leq \min(M_0 + M_1, L_1)$.

Si $m_0 + m_1 \geq l_1$ on définit

$$\begin{cases} m'_0 = m_0, \\ m'_1 = m_1 \end{cases} \quad \text{et} \quad \begin{cases} m''_0 = e, \\ m''_1 = f, \end{cases} \quad \text{avec } \begin{cases} m_0 \leq e \leq M_0, \\ m_1 \leq f \leq M_1 \end{cases} \quad \text{et } e+f \leq L_1.$$

Donc $l_1 \leq m_0 + m_1 \leq e+f \leq \min(M_0 + M_1, L_1)$.

On considère la relation

$$(2) \quad \max(m_0 + m_1, l_1) \leq \min(M_0 + M_1, L_1).$$

On définit les quantités

$$\begin{cases} m'_{i+1} = \max(m_{i+1}, m'_i + l_{i+1}), \\ m''_{i+1} = \min(M_{i+1}, m''_i + L_{i+1}), \end{cases} \quad \text{pour } i=1, 2, \dots, p-1$$

et on considère la relation

$$(3) \quad \max \left(\sum_{i=1}^p l_i, m_0 + m_p \right) \leq \min \left(\sum_{i=1}^p L_i, M_0 + M_p \right).$$

Remarque 1. Tout comme dans le cas [3] il en résulte

$$\begin{cases} m_i \leq m'_i, & \text{pour } i=0, 1, \dots, p \text{ et } m'_i \leq m'_{i+1}, & \text{pour } i=1, 2, \dots, p-1. \\ m'_i \leq M_i, & \end{cases}$$

Remarque 2. Généralement les conditions

$$(4) \quad m'_i \leq m''_i, \quad \text{pour } i=2, 3, \dots, p,$$

ne sont pas satisfaites.

Théorème. La condition nécessaire et suffisante que dans un graphe éventail existe une circulation en satisfaisant (1), de la forme

$$(5) \quad F(R_1) = F(P_0) + F(P_1),$$

$$(6) \quad F(P_{i+1}) = F(P_i) + F(R_{i+1}) \quad \text{pour } i=1, 2, \dots, p-1,$$

où le sens de la circulation est spécifié sur le graphe donné, est que les conditions (2), (3) et (4) soient satisfaites.

Nécessité. Du (1) nous avons

$$l_1 \leq F(R_1) \leq L_1,$$

$$m_0 + m_1 \leq F(P_0) + F(P_1) \leq M_0 + M_1$$

et du (5) il résulte (2).

Du (5) et du (6) il y a $F(P_0) + F(P_p) = \sum_{i=1}^p F(R_i)$. Donc

$$m_0 + m_p \leq F(P_0) + F(P_p) \leq M_0 + M_p,$$

$$\sum_{i=1}^p l_i \leq \sum_{i=1}^p F(R_i) \leq \sum_{i=1}^p L_i;$$

de ici il résulte (3).

Nous avons de cette construction

$$m_0 \leq m'_0 \leq F(P_0) \leq m''_0 \leq M_0,$$

$$m_1 \leq m'_1 \leq F(P_1) \leq m''_1 \leq M_1,$$

$$l_1 \leq F(R_1) \leq L_1$$

et supposant $m'_i \leq F(P_i) \leq m''_i$ ensemble avec

$$l_{i+1} \leq F(R_{i+1}) \leq L_{i+1},$$

$$m_{i+1} \leq F(P_{i+1}) \leq M_{i+1}$$

et parce que $F(P_{i+1}) = F(P_i) + F(R_{i+1})$ résulte (4).

Suffisance. On définit

$$F(P_p) = s_p \quad \text{avec} \quad m'_p \leq s_p \leq m''_p,$$

$$F(P_i) = \max(m'_i, s_{i+1} - L_{i+1}) = s_i, \quad \text{pour} \quad i = p-1, p-2, \dots, 1;$$

$$F(R_{i+1}) = \begin{cases} s_{i+1} - m'_i & \text{si } m'_i \geq s_{i+1} - L_{i+1}, \\ L_{i+1} & \text{si } m'_i < s_{i+1} - L_{i+1}, \end{cases} \quad \text{pour } i = p-1, p-2, \dots, 1.$$

Si $m'_i \geq s_{i+1} - L_{i+1}$ nous avons $F(P_i) = m'_i$ et $F(R_{i+1}) = s_{i+1} - m'_i \leq L_{i+1}$. Du

$$\left. \begin{array}{l} m'_{i+1} \leq s_{i+1} \\ m'_i + l_{i+1} \leq m'_{i+1} \end{array} \right\} \Rightarrow l_{i+1} \leq m'_{i+1} - m'_i \leq s_{i+1} - m'_i.$$

Donc $l_{i+1} \leq F(R_{i+1}) \leq L_{i+1}$.

Si $m'_i < s_{i+1} - L_{i+1}$ nous avons $F(P_i) = s_{i+1} - L_{i+1}$ avec $m'_i \leq F(P_i) \leq m''_i$, parce que $m'_{i+1} \leq s_{i+1} \leq m''_{i+1} \leq m'_i + L_{i+1} \Rightarrow s_{i+1} - L_{i+1} \leq m'_i$ et $F(R_{i+1}) = L_{i+1}$.

On vérifie les conditions (6).

Similaire de $m'_1 \leq F(P_1) = s_1 \leq m''_1$, on définit $F(P_0) = m'_0$ et $F(R_1) = m'_0 + s_1$.

Remarque 3. Si on fait que les arcs extrêmes P_0 et P_p coïncident dans l'arc P_{0p} prenant $l_{0p} = m_0 + m_p$ et $L_{0p} = M_0 + M_p$, on obtient un graphe éventail fermé [4] (fig. 2). Avec un circulation du type

$$F(R_1) = F(P_0) + F(P_1),$$

$$F(P_{i+1}) = F(P_i) + F(R_{i+1}), \quad \text{pour } i = 1, 2, \dots, p-1.$$

$$F(P_{0p}) = F(P_0) + F(P_p),$$

Réciproquement, une circulation du graphe éventail fermé ci-dessus peut être réduite à une circulation d'un graphe éventail du type du théorème démontré, en scindant l'arc P_{0p} dans les arcs extrêmes P'_0 et P'_p en prenant

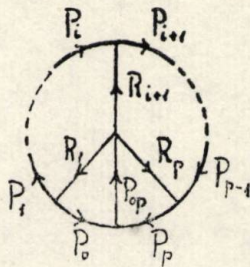


Fig. 2

$$m'_0 = m_0 \quad \text{et} \quad m'_p = m_p, \quad \text{si } m_0 + m_p \geq m_{0p},$$

$$m'_0 = m_0 \quad \text{et} \quad m'_p = m_{0p} - m_0, \quad \text{si } m_0 + m_p < m_{0p},$$

$$m''_0 = M_0 \quad \text{et} \quad m''_p = M_p, \quad \text{si } M_0 + M_p \leq M_{0p},$$

$$m''_p = m'_p \quad \text{et} \quad m''_0 = M_{0p} - m'_p, \quad \text{si } M_0 + M_p > M_{0p}.$$

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TIPURI DE CIRCULAȚII ÎN GRAFURI EVANTAI (III)

(Rezumat)

Se studiază problema circulației de forma (1) într-un graf evantai, cind sint satisfăcute condițiile de mărginire (2), (3) și (4). Se obțin condiții necesare și suficiente pentru existența unei circulații admisibile în grafuri de acest tip.

Se face apoi legătura între circulația din grafurile evantai și circulația din grafurile evantai închise cu condițiile de mărginire adaptate convenabil.



Résumé. On étudie le problème de la circulation de la forme (1) dans un graphe étoilé, quand sont satisfaites les conditions de bornage (2), (3) et (4). On obtient des conditions nécessaires et suffisantes pour l'existence d'une circulation admissible dans des graphes de ce type. On fait ensuite le lien entre la circulation dans les graphes étoilés et la circulation dans les graphes étoilés fermés avec des conditions de bornage adaptées convenablement.

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RECENZII

BOOK REVIEWS

РЕЦЕНЗИИ

G. S. ASANOV, *Finsler Geometry. Relativity and Gauge Theories*. D. Reidel Publishing Company, Dordrecht, 1985, X + 370 p.

The purpose of the book is to elaborate a systematic study of the main techniques for extending general relativity based on the geometry of Finsler spaces. The first two chapters stand for a rapidly introduction to Finsler geometry. The Chapter 4 is concerned with the general theory of invariance identities and its applications to constructions of connection coefficients and fundamental tensor densities. In Chapters 3 and 5 there are formulated and studied two Finslerian approaches to obtain generalizations of the physical field equations. The first approach is based on the concept of osculation, according to which a vector field $y^i(x)$ is going to be substituted for the directional variable y^i any time when a Lagrangean density is constructed. The second approach is based on the notion of indicatrix in a Finsler space, which is in fact a hypersurface in the tangent space to a Finsler space, defined by the equation $F(x, y) = 1$, where F is the Finslerian metric function. Some problems of classical mechanics are studied in Chapter 6 from the Finslerian viewpoint. Chapter 7 is concerned with Finslerian extension of the special principle of relativity and Finslerian kinematics.

There are added two appendixes. In the first one the author describes the Rund theory with respect to the concept of direction-dependent connection. The second one is dealing with a generalization of both, the Finslerian gauge ideas from Chapter 5 and from the first appendix. Finally the author provides to the reader an up to day list of publication on Finsler geometry.

We conclude by saying that Asanov's book is an important contribution to literature and should benefit both experts and novices in applications of Finslerian geometry.

Aurel Bejancu

SCHULTZ-PISZACHICH W., *Nonlinear Models of Flow, Diffusion and Turbulence*, Teubner-Texte zur Physik, Band 6, Leipzig, 1985, XVI + 204 p.

This book is devoted to the theory of self-consistent field with applications to electric space charge distributions, mainly to vortex and turbulent flows. It manages to reach scientific engineers as well as theoretical physicists. Such, the foundation of Hamilton's classical theory is represented for engineers and some fundamental principles of hydrodynamics are memorized for physicists.

There are also studied, vortices and eddies in an averaged stationary turbulent flow, one of them as a solution of a statistical vortex equation and as a new exact solution of Navier-Stokes equations.

A class of homogeneous-isotropic turbulence fields is calculated in great detail. In the case of the field between rotating cylinders and the turbulent flow in tubes new aspects are shown. The velocity profile of Poiseuille flow in tubes for large Reynolds numbers, is fully in agreement with that resulting from measurements.

With the mathematical supplements from appendices, the book can be used by researchers and graduate students of fluid mechanics in its widest sense.

C. Fetecău

GEORGHE M. RASSIAS (Ed.), *Differential Topology-Geometry and Related Fields, and Their Applications to the Physical Sciences and Engineering*. Dedicated to Henry Poincaré (1854–1912), TEUBNER-TEXTE zur Mathematik, Band 76, Teubner Verlagsgesellschaft, Leipzig, 1985.

The book is dedicated to the memory of Henry Poincaré on the occasion of his 130-th anniversary and contains a series of twenty invited research and expository papers. Written by internationally recognized authorities the book provides an insight and analysis on recent new theories in the fields of Differential Topology, Global Analysis of Differential Equations, and Calculus of Variation, with their Applications to the Physical Sciences and Engineering.

The book is of great interest for anyone working in mathematics education, philosophy of mathematics, in applied research, and technology.

Gh. Gr. Ciobanu

WOLFGANG NATHER, *Effective Observation of Random Fields*. TEUBNER-TEXTE, Band 72, Teubner Verlagsgesellschaft, Leipzig, 1985.

The main purpose of the book is to find designing methods for models with correlated observations which lead to the consideration of stochastic processes and random fields. Random fields with linear trend are considered. For such kind of models the possibility to extend the known designing methods for linear models with uncorrelated observations is firstly explored, and secondly the approximative designing methods are presented.

The first three chapters contain introductory results, and the results concerning the effective observations of random fields are presented in the other seven chapters of the book.

The book is well written and is of interest for mathematicians and engineers.

Gh. Gr. Ciobanu



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